Math 5C Spring 2010
Practice Final Exam

Name____________________________________

Perm No.________________________________

M. Choice	
F. Resp. 1	
F. Resp. 2	
F. Resp. 3	
F. Resp. 4	
F. Resp. 5	
F. Resp. 6	
F. Resp. 7	
F. Resp. 8	
F. Resp. 9	
F. Resp. 10	
F. Resp. 11	
F. Resp. 12	
Total	

Directions:

1. There are 340 points on this exam; 250 points = 100%.
2. Each multiple choice problem is 5 points.
3. Each multiple choice problem has exactly one best answer.
4. No multiple choice problem requires heavy computation.
5. Each free response problems is 20 points.
6. Free response questions require justification; no work, no credit.
7. **A blank free-response problem is awarded 5 points.**
8. You may use any hand-written notes or notes I've typed.
9. No books or electronic devices are allowed.

Multiple Choice

1. Let S be the sphere of radius R centered at the origin in $\mathbb{R}^3$. Evaluate $\displaystyle\iint_S x^2\,d\sigma$.
   (a) $4\pi R^2$
   (b) $4\pi R^4$
   (c) $4\pi R^2/3$
   (d) $4\pi R^4/3$
   (e) None of the above

2. For which real numbers α does this series converge?
$$\sum \left(\sin\left(\frac{1}{n}\right) - \frac{1}{n}\right)^{\alpha}$$
   (a) $\alpha > 1/3$
   (b) $\alpha < 1/3$
   (c) $\alpha > 1$
   (d) $\alpha < 1$
   (e) None of the above

3. If we rewrite
$$\int_0^1 \int_0^y \int_0^y f(x,y,z)\,dz\,dx\,dy$$
using the order $dx\,dy\,dz$, what is the upper limit on x?
   (a) 1
   (b) y
   (c) $1 - z$
   (d) $1 - y$
   (e) None of the above

4. What is the radius of convergence of $\sum 3^n x^n n!/n^n$?
   (a) 1
   (b) 3
   (c) 1/3
   (d) 0
   (e) ∞

5. Let S be a surface in $\mathbb{R}^3$. The unit binormal to the curve ∂S is the cross product of the unit tangent and unit normal vectors: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Evaluate
$$\oint_{\partial S} \mathbf{B} \cdot d\mathbf{r}$$
   (a) 0
   (b) 2π
   (c) area(S)
   (d) length(∂S)
   (e) None of the above

6. Which of these series is smaller than $\sum 1/(1+n^2)$?

   (a) $\sum 1/n^2$
   (b) $\sum \arctan n$
   (c) $\sum 1/n^3$
   (d) $\sum 1/(1+n)$
   (e) $\sum (1+n^{-1})/n^2$

7. Suppose that $f(x)$ is a 2π-periodic function on $\mathbb{R}$ and that $f(x) = xe^x$ for $-\pi \leq x \leq \pi$. If $g(x)$ is the Fourier series of f, what is $g(\pi)$?

   (a) πe^{π}
   (b) $-\pi e^{-\pi}$
   (c) $\pi \sinh \pi$
   (d) $\pi \cosh \pi$
   (e) None of the above

8. Let S be the unit sphere centered at $(0, 0, 1)$. In spherical coordinates, which of the following equations describes S?

   (a) $\rho = 1$
   (b) $\rho = \cos \phi$
   (c) $\rho = 2 \cos \phi$
   (d) $\rho = \sin \phi$
   (e) $\rho = 2 \sin \phi$

9. Using the inner product for functions on $[-\pi, \pi]$, what is the angle between 1 and x^2?

   (a) 0
   (b) $\arccos(2\pi^3/3)$
   (c) $\arccos(\sqrt{5}/3)$
   (d) $\pi/2$
   (e) None of the above

10. Evaluate $\displaystyle\int_{-\pi}^{\pi} \sinh 17ix \cosh 13ix \, dx$.

    (a) 0
    (b) $i\pi$
    (c) $4i\pi$
    (d) $30i\pi$
    (e) None of the above

11. Find $f^{(10)}(0)$ for $f(x) = \arctan(x^2)$.

(a) 0
(b) 10!
(c) 10!/10
(d) $-10!/10$
(e) None of the above

12. If we use

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

how many terms do we need to approximate 4 digits (after the decimal point) of $\ln 2$ according to the alternating series test?

(a) Around 20
(b) Around 200
(c) Around 2,000
(d) Around 20,000
(e) Around 200,000

13. Sum the double series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{5^{m+n}}$$

(a) 5/4
(b) 25/4
(c) 5/2
(d) Diverges
(e) None of the above

14. If the fourth term of a geometric series is 1/3! while the fifth term is 1/4!, what is the sum of the series?

(a) 128/9
(b) 512/9
(c) e
(d) π
(e) None of the above

15. Which of these functions is harmonic in $\mathbb{R}^2$?

(a) $\cos x \sin y$
(b) $\cosh x \sinh y$
(c) $\ln(x + y)$
(d) $e^x \cosh y$
(e) $\cos x \cosh y$

16. Which of these integrals gives the volume of the unit ball B in $\mathbb{R}^3$ centered at the origin?

    (a) $\iiint_B \rho^2 \sin\phi \, dV$
    (b) $\iiint_B (x^2+y^2+z^2)\, dV$
    (c) $\int_0^{2\pi}\int_0^{\pi}\int_0^1 d\rho\, d\phi\, d\theta$
    (d) $\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\, dr\, d\theta\, d\phi$
    (e) None of the above

17. Using $I(a) = \int_0^1 \frac{x^a - 1}{\ln x}\, dx$, what is $\int_0^1 \frac{x-1}{\ln x}\, dx$?

    (a) 0
    (b) $\ln 2$
    (c) $\ln 3$
    (d) $\ln 4$
    (e) None of the above

18. Which of the following is a sufficient reason for swapping an infinite sum and integral?

    (a) The terms are all positive
    (b) The sum is finite
    (c) The integral is finite
    (d) They are next to each other; they have to be switched
    (e) We just felt like it

19. Recall the Cauchy condensation test: for a positive decreasing sequence, $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges. Which of the following converge?

    (a) $\sum 1/n^{1+1/n}$
    (b) $\sum \log_2 n/(\log_2\log_2 n)^2$
    (c) $\sum (n\log_2^2 n)^{-1}$
    (d) $\sum (\log_2\log_2\log_2 n)^{-3}$
    (e) None of the above

20. (Bonus Problem) On the island of knights and knaves, knights always tell the truth and knaves always lie. Five people—named A, B, C, D, and E—live on this island and make the statements below. Choose one person you are sure to be a knight.

    (a) E could say B is a knight.
    (b) D and myself are different types.
    (c) D is a knave.
    (d) Either I'm a knight or E is a knave (possibly both)
    (e) Either A or C is a knave (possibly both)

Free Response

1. Define the curve C as $\mathbf{r}(t) = (\cos^3 t, \sin^3 t)$, $0 \le t \le 2\pi$. Define the curvature of C to be

$$\kappa(t) = \frac{1}{\|\mathbf{r}'(t)\|} \left\| \frac{d}{dt} \left(\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \right) \right\|.$$

Evaluate $\displaystyle\int_C \kappa \, ds$.

2. Evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx.$

3. Evaluate $\displaystyle\iiint_{\mathbb{R}^3} \frac{e^{-z^2}}{1+(x^2+y^2)^2}\, dV$.

4. Let f and g be smooth scalar functions and B a ball in $\mathbb{R}^3$. Show that

$$\iint_{\partial B} g\nabla f \cdot d\mathbf{A} = \iiint_B (g\Delta f + \nabla f \cdot \nabla g)\, dV.$$

5. Let B be a ball in $\mathbb{R}^3$. Suppose that f and g are scalar functions on $\mathbb{R}^3$ such that $f = g$ on ∂B. If f is harmonic, prove the following energy functional inequality:

$$\iiint_B \|\nabla f\|^2 \, dV \leq \iiint_B \|\nabla g\|^2 \, dV.$$

You can assume the result of the previous problem and the Cauchy inequality:

$$\iiint_B \nabla f \cdot \nabla g \, dV \leq \left(\iiint_B \|\nabla f\|^2 \, dV \right)^{1/2} \left(\iiint_B \|\nabla g\|^2 \, dV \right)^{1/2}$$

6. Let S be the upper half ($z \geq 0$) of the unit sphere centered at the origin in $\mathbb{R}^3$, oriented upward. Define $\mathbf{F}(x, y, z) = (x, \ln(1 + y^4), z)$ and $g(x, y, z) = xy^2 e^z \cos(xy) \sin(z)$. Use integration by parts to evaluate

$$\iint_S \nabla g \times \mathbf{F} \cdot d\mathbf{A}.$$

7. Using the trigonometric identity

$$\tan x = \cot x - 2\cot 2x,$$

evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan\left(\frac{1}{2^n}\right)$$

8. Determine, with justification, whether this series converges.

$$\sum n^{1/n}\left(1-\frac{1}{n}\right)^{n^2}$$

9. Newton's binomial theorem states that for any real number α and $|t| < 1$,

$$(1+t)^\alpha = 1 + \alpha t + O(t^2).$$

Use this (or find another way) to evaluate $\lim_{n\to\infty} \sin^2(\pi\sqrt{n^2+n})$.

10. For polynomial functions defined on $[-1,1]$ we have the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

The Legendre polynomials

$$p_0 = 1, \quad p_1 = x, \quad p_2 = \frac{1}{2}(3x^2 - 1), \quad p_3 = \frac{1}{2}(5x^3 - 3x), \ldots$$

form an orthogonal basis. Find the coefficient c_2 in the basis expansion

$$x^{100} = \sum_{n=0}^{\infty} c_n p_n(x)$$

11. Let R be the unbounded region in $\mathbb{R}^2$ with $x \geq 0$ and $0 \leq y \leq \pi$. Find all harmonic functions $u(x, y)$ in R which can be written in the form $u = F(x)G(y)$ and satisfy

$$\begin{aligned} u(x,0) &= 0 \qquad && \text{for all } x > 0 \\ u(x,\pi) &= 0 \qquad && \text{for all } x > 0 \\ \lim_{x\to\infty} u(x,y) &= 0 \qquad && \text{for all } 0 \leq y \leq \pi \end{aligned}$$

12. Define the functions for $s > 1$

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad \text{and} \qquad \Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx.$$

Using a series expansion, show that for all real numbers $s > 1$,

$$\int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx = \zeta(s)\Gamma(s).$$

Justify the interchange of sum and integral somehow. (This integral is very important in statistical physics)