Math 5C Quiz Solutions

1. Let C be the unit circle centered at the origin in $\mathbb{R}^2$, oriented counterclockwise. Evaluate $\displaystyle\int_C 2\,ds$.

   Solution. $\displaystyle\int_C 2\,ds = 2\,\text{length}(C) = 4\pi.$ ☐

2. Evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{4-y^2}} x^2\sqrt{x^2+y^2}\,dx\,dy$.

   Solution. Switching to polar gives $\displaystyle\int_0^{\pi/2}\int_0^2 r^4\cos^2\theta\,dr\,d\theta = \frac{8\pi}{5}.$ ☐

3. Let S be the unit sphere centered at the origin in $\mathbb{R}^3$. Evaluate $\displaystyle\iint_S x^2\,d\sigma$.

   Solution. On S, $x^2+y^2+z^2=1$. Hence $\displaystyle\iint_S (x^2+y^2+z^2)\,d\sigma = \iint_S d\sigma = \text{area}(S) = 4\pi$. By symmetry, $\displaystyle\iint_S x^2\,d\sigma = \iint_S y^2\,d\sigma = \iint_S z^2\,d\sigma$, so the desired integral is $4\pi/3$. ☐

4. Let C be given by $\mathbf{r}(t) = \left(\dfrac{\cos^7 t\sin^9 t}{1+t^2}, \ln(1+\arctan t)\right)$, $t>0$. Given $f(x,y)=x^3+y^3$, evaluate $\displaystyle\int_C \nabla f\cdot d\mathbf{r}$.

   Solution. $\displaystyle\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(\infty)) - f(\mathbf{r}(0)) = f(0,\ln(1+\pi/2)) - f(0,0) = \ln^3(1+\pi/2).$ ☐

5. Let S be the surface given by $z = 1-x^2-y^2$, $z\geq 0$, oriented upward. If $\mathbf{F} = (xz^2, \ln(1+x^2), 1)$, evaluate

   $$\iint_S \mathbf{F}\cdot d\mathbf{A}$$

   Solution. Let R be the solid region between S and the plane $z=0$. The boundary of R consists of S and a unit disk in the plane $z=0$, oriented downward. On the disk, $\mathbf{F} = (0,\ln(1+x^2),1)$ and $\mathbf{n}=(0,0,-1)$, so $\mathbf{F}\cdot d\mathbf{A} = -d\sigma$ everywhere on the disk. The flux through the disk is $-\pi$, so the divergence theorem gives

   $$-\pi + \iint_S \mathbf{F}\cdot d\mathbf{A} = \iiint_R \nabla\cdot\mathbf{F}\,dV = \iiint_R z^2\,dV.$$

   In cylindrical coordinates,

   $$\iint_S \mathbf{F}\cdot d\mathbf{A} = \pi + \iiint_R z^2\,dV = \pi + \int_0^{2\pi}\int_0^1\int_0^{1-r^2} rz^2\,dz\,dr\,d\theta = \pi + \frac{2\pi}{3}\int_0^1 r(1-r^2)^3\,dr = \frac{13\pi}{12}$$ ☐

6. A sequence satisfies

$$\sum_{n=1}^{N} a_n = \frac{\arctan(N^{-N})}{5^N} + \frac{1}{N} + 2^{-N}\sqrt{\ln N} + 1$$

for all positive integers N. Evaluate $\lim_{n\to\infty} a_n$.

Solution. Taking $N \to \infty$ gives $\sum_1^\infty a_n = 1$. Since the series converges, the sequence converges to 0. □

7. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{1+n^2}$. What is $f^{(100)}(0)$?

Solution. $f^{(100)}(0)$ is the coefficient of x^{100} times 100!. That is, $f^{(100)}(0) = 100!/(1+100^2)$. □

8. Evaluate $\lim_{x\to 0} \frac{x^2 + \ln\cos^2 x}{x^4}$.

Solution. For small t we have $\ln(1+t) = t - t^2/2 + O(t^3)$ and $\cos t = 1 - t^2/2 + t^4/24 + O(t^6)$. Putting it together,

$$\begin{aligned}
\ln\cos^2 x &= 2\ln\cos x \\
&= 2\ln\left[1 + \left(-\frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)\right] \\
&= 2\left[\left(-\frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)^2 + O(x^6)\right] \\
&= -x^2 - \frac{x^4}{6} + O(x^6).
\end{aligned}$$

This gives the limit:

$$\lim_{x\to 0} \frac{x^2 + \ln\cos^2 x}{x^4} = \lim_{x\to 0} \frac{-x^4/6 + O(x^6)}{x^4} = \lim_{x\to 0}\left(-\frac{1}{6} + O(x^2)\right) = -\frac{1}{6}$$

□

9. Define $f(x) = \sum_{n=1}^{\infty}\left(e^{-n^2}\sin(nx) + \sin(1/n^3)\cos(nx)\right)$. Evaluate $\int_{-\pi}^{\pi} f(x)\sin 12x\,dx$.

Solution. Assuming uniform convergence and all that,

$$\int_{-\pi}^{\pi} f(x)\sin 12x\,dx = e^{-144}\int_{-\pi}^{\pi}\sin^2(12x)\,dx = \pi e^{-144}.$$

□