Vector Calculus Notation

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Generally we use lower case letters f, g, h for scalar functions and capital F, G, H for vector fields. When typed, vector fields are often boldfaced.

Derivatives

Our major derivative is

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Using this 'vector' we an construct three major derivatives:

1. Gradient of a scalar function.

$$\nabla f = \mathrm{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

2. Divergence of a vector field.

$$\mathrm{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z},$$

   where we denote $\mathbf{F} = (f_1, f_2, f_3)$.

3. Curl of a vector field.

$$\nabla \times \mathbf{F} = \mathrm{curl}\mathbf{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

   where we denote $\mathbf{F} = (f_1, f_2, f_3)$ and $\partial_x = \partial / \partial x$.

4. Laplacian of a scalar function.

$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Integrals

1. Line/path integrals. Let C be a curve, parametrized by $\mathbf{r}(t)$ with $a \leq t \leq b$. We denote by ds an infinitesimal piece of arclength along C (Other names include 'arclength measure on C' or 'curvilinear length'). The line integral of a scalar function f is defined by

$$\int_C f \, ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt.$$

   If instead of the legnth ds we consider a tiny vector along the curve of length ds, we have the infinitesimal vector $d\mathbf{r}$. The vector path integral (or work integral, or circulation) of a vector field $\mathbf{F}$ is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt.$$

2. Area integrals. Let R be a region in $\mathbb{R}^2$. A tiny piece of area of R is denoted by dA, leading to the area integral of scalar functions

$$\int_R f \, dA.$$

   This can usually be evaluated via Fubini's theorem, which isn't simply part of the definition of the intgeral.

3. Volume integrals. Let R be a region in $\mathbb{R}^3$. A tiny piece of volume of R is denoted by dV, leading to the volume integral of scalar functions

$$\int_R f\,dV.$$

This can usually be evaluated via Fubini's theorem, which isn't simply part of the definition of the intgeral.

4. Surface integrals. Let S be a surface, parametrized by $\mathbf{r}(u,v)$, with (u,v) belonging to some parameter set $D \subseteq \mathbb{R}^2$. We denote by $d\sigma$ an infinitesimal piece of area on the surface (Other names include 'surface measure on S' or 'curvilinear area'). The surface integral of a scalar function f is defined by

$$\int_S f\,d\sigma = \int_D f(\mathbf{r}(u,v))\|r_u \times r_v\|\,dA(u,v).$$

Note that the right side is a (flat) area integral in $\mathbb{R}^2$. The notation $\mathbf{r}_u$ denotes a partial derivative with respect to the parameter u.

If instead of the area $d\sigma$ we consider a tiny normal vector out of the surface of length $d\sigma$, we have the infinitesimal vector $d\mathbf{A}$. The vector surface integral (or flux integral) of a vector field $\mathbf{F}$ is

$$\int_S \mathbf{F}\cdot d\mathbf{A} = \int_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA(u,v).$$