# Interesting Problems

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1. One hundred balls are placed into one hundred boxes. Suppose that for each $k = 1, 2, \dots, 99$ no set of $k$ boxes contains exactly $k$ balls. Prove that all balls are in the same box.

2. Two linear operators $P, Q$ on a Hilbert space satisfy $[P, Q] = i\hbar I$, where $\hbar \in \mathbb{R} \setminus \{0\}$ and $[P, Q] = PQ - QP$. Prove that the underlying space is infinte-dimensional and that at least one of the operators is unbounded. Hint: consider $[P, Q^n]$.

3. Let $n \in \mathbb{N}$. Evaluate $\displaystyle\sum_{j=0}^{n} \binom{2n}{2j} (-3)^j$.

4. The numbers $a$ and $b$ are randomly chosen independently and uniformly from the interval $[-1, 1]$. Find the probability that $a^{2/3} + b^{2/3} \leq 1$.

5. Let $n$ be a positive integer. Prove that $\dfrac{(n!)!}{(n!)^{(n-1)!}}$ is an integer as well.

6. For a positive integer $n$ define $[n] = \{1, 2, \dots, n\}$. Given $S \subseteq [n]$, define $\omega(S) = \max S - \min S$. Find the average value of $\omega$ over all subsets of $[n]$.

7. Fix $n \in \mathbb{N}$ and let $S \subset \mathbb{R}$ be a set with $n$ elements. For every nonempty subset of $S$ one can compute an arithmetic mean of its elements; thus we can form $2^n - 1$ means. Show that the arithmetic mean of these $2^n - 1$ numbers is the arithmetic mean of $S$.

8. Two people buy identical cups of coffee at the same (high) temperature. The woman adds cold milk immediately, whereas the man waits for 5 minutes before adding milk. The milk in both cases is the same amount and cold. Each person consumes their coffee 6 minutes after drinking it. Who drinks the hotter coffee?

9. Find the number of order $k$-tuples of sets $(A_1, A_2, \dots, A_k)$ such that both the following hold:
$$A_1 \cup A_2 \cup \cdots \cup A_k = \{1, 2, 3, \dots, n\},$$
$$A_1 \cap A_2 \cap \cdots \cap A_k = \varnothing.$$

10. Let $n$ be a positive integer. For a permutation $\pi$ of $\{1, 2, \dots, n\}$ define
$$f(\pi) = \sum_{k=1}^{n} |k - \pi(k)|.$$
Find the average value of $f$ over all permutations.

11. Let $n$ be a positive integer and $X$ be a set with $n$ elements. Find the sum
$$\sum_{A,B \subseteq X} |A \cap B|.$$

12. For $a, b, c > 0$, prove that
$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}.$$

13. Solve the following system of equations in real numbers:

$$\begin{aligned} 2008\log\lfloor x\rfloor + \{\log y\} &= 0 \\ 2008\log\lfloor y\rfloor + \{\log z\} &= 0 \\ 2008\log\lfloor z\rfloor + \{\log x\} &= 0, \end{aligned}$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer function and $\{\cdot\}$ denotes the fractional part.

14. Find all nonnegative integer solutions to $1! + 2! + \cdots + x! = y^2$.

15. Find all nonnegative integer solutions to $x! + y! = x^y$.

16. Let $a, b$ be relatively prime positive integers. Prove that $a^{\phi(b)} + b^{\phi(a)} \equiv 1 \pmod{ab}$, where $\phi$ is Euler's phi function.

17. A projectile is launched on earth such that at its highest point, its potential energy is twice its kinetic energy. At what angle was the projectile fired?

18. I was once asked by a student why $22/7 \neq \pi$. Answer the question using

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx.$$

19. Evaluate

$$\sum_{\gcd(m,n)=1} \frac{1}{m^2n^2},$$

where the series sums over all pairs of relatively prime positive integers $(m,n)$.

20. Let $f : [0,4] \to \mathbb{R}$ be differentiable on $(0,4)$. Prove that there exists $c \in (0,4)$ so that

$$f'(c) < 1 + f(c)^2.$$

21. Let $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. Sum the series

$$\sum_{n=1}^{\infty} \frac{1}{F_{n+2}F_n}.$$

22. Show that $\dfrac{1+\sqrt{5}}{2} < \dfrac{\pi^2}{6}$ (by hand!).

23. Show that $\dfrac{1}{\log_2 \pi} + \dfrac{1}{\log_5 \pi} > 2$ (by hand!).

24. Show that $e + \ln 4 > 4$ (by hand!).

25. Evaluate $\lim\limits_{n\to\infty} n\sin(2\pi n!e)$.

26. Evaluate

$$\lim_{n\to\infty} n^2\left(\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} - e\right).$$

27. Fix $n \geq 3$ and $\omega = \exp(2\pi i/n)$. Evaluate $\displaystyle\sum_{A\subseteq[n-1]} \cos\left(\sum_{i\in A} \pi\omega^i\right)$, where $[x] = \{1,2,3,\ldots,x\}$.

28. The function $W(x)$ satisfies $W(x)e^{W(x)} = x$ for all $x$. Evaluate

$$\int_0^e W(x)\,dx.$$

29. For $a > 0$, let $f_a(x) = a^x$. Let $\ell_1$ be the tangent line to the graph $f_a$ at $x = 0$. Let $\ell_2$ be the tangent line to the graph of $f_a$ which passes through the origin. The lines $\ell_1$ and $\ell_2$ intersect, forming an acute angle $\theta$. For which $a$ is $\theta$ maximized?

30. Let $f$ be a function such that $\displaystyle\int_0^\pi f(\cos^2 x)\,dx = I$. Evaluate $\displaystyle\int_0^\pi x f(\cos^2 x)\,dx$.

31. Let $a, b > 0$. Evaluate
$$\int_0^\infty \left(e^{-a^2/x^2} - e^{-b^2/x^2}\right) dx.$$

32. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $0 < a < b$ are fixed numbers. For an integer $n \geq 2$ and a real number $\delta > 0$, define
$$D_\delta = \{x \in \mathbb{R}^n : a\delta \leq |x| \leq b\delta\}.$$
Evaluate the limit
$$\lim_{\delta \to 0^+} \int_{D_\delta} \frac{f(|x|)}{|x|^n}\,dx.$$

33. Each point in space is colored either red or blue. Show that there is either a unit square with three red vertices or a unit square with four blue vertices.

34. The positive integers are colored black and white. Given two differently colored numbers, their sum is black and their product is white. Prove that the product of white numbers is white. Find all possible colorings.

35. Of the 1985 people in attendance at a recent international math conference, no one spoke more than 5 languages and at least 2 people in every set of 3 attendees spoke a common language. Prove that some language was spoken by at least 200 people at the meeting.

36. Prove that there is a way to color the points of $\mathbb{Q}^2$ red and blue so that no two points of the same color are exactly one unit apart.

37. Consider a paper punch that can be centered at any point in the plane and that, when operated, removes from the plane precisely those points whose distance to the center is irrational. How many punches are required to remove every point?

38. Let $A$ be an $n \times n$ symmetric positive definite matrix over the real numbers. Evaluate
$$\int_{\mathbb{R}^n} \exp(-\langle \mathbf{x}, A\mathbf{x}\rangle)\,d\mathbf{x},$$
where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$.