Math 8, Summer 2012
Exam 1 Solutions

Short Answer

1. Given sets A and B, give a precise definition of $A \subseteq B$.

   For every $x \in A$ we have $x \in B$.

2. Let $f : A \to B$ be a function and $S \subseteq B$. Give a precise definition of $f^{-1}(S)$.

   $f^{-1}(S) = \{x \in A : f(x) \in S\}$

3. A sequence of continuous functions $f_1, f_2, f_3 \ldots$, each mapping from $[0, 1]$ into $\mathbb{R}$, is said to *converge uniformly* if and only if:

   For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for all integers $n, m \geq N$ and $x \in [0, 1]$ we have $|f_n(x) - f_m(x)| < \epsilon$.

   Give a precise statement of what it means for such a sequence not to converge uniformly.

   There is an $\epsilon > 0$ so that for any positive integer N there are integers $m, n \geq N$ and a real number $x \in [0, 1]$ so that $|f_n(x) - f_m(x)| \geq \epsilon$.

4. Precisely define what it means for the function $f : A \to B$ to be surjective.

   For every $y \in B$ there is $x \in A$ so that $f(x) = y$.

5. Given a collection of sets $\{A_i : i \in I\}$, precisely define $\bigcup_{i \in I} A_i$.

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

6. Which of these is not bijective?

   (a) The identity map $\mathbb{Z} \to \mathbb{Z}$
   (b) A 90° rotation of $\mathbb{R}^2$ about the origin
   (c) A translation of $\mathbb{R}^3$ by 3 units along an axis
   (d) The inclusion map $\mathbb{N} \hookrightarrow \mathbb{Z}$
   (e) None of the above

   Choice (d), since the inclusion map is not surjective.

7. Given sets A and B, precisely define $A \times B$.

   $A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$

8. Give a precise definition of what it means for a real number x to be rational.

   There exist integers m, n with $n > 0$ so that $x = m/n$.

Problems

1. Let a and b be integers. Prove that $a+b$ is even if and only if a^2+b^2 is even.

   Proof. ($\Rightarrow$) Suppose that $a+b$ is even. Find an integer k so that $a+b=2k$; then

   $$a^2+b^2=(a+b)^2-2ab=2\cdot(2k^2-ab),$$

   an even integer.
   ($\Leftarrow$) Suppose that $a+b$ is odd. Find an integer ℓ so that $a+b=2\ell+1$; then

   $$a^2+b^2=(a+b)^2-2ab=2\cdot(2\ell^2+2\ell-ab)+1,$$

   an odd integer. $\square$

2. Let A and B be sets with $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$. Prove that either $A \subseteq B$ or $B \subseteq A$.

   Proof. Assume that $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ and $A \not\subseteq B$. Then there is an $x \in A$ so that $x \notin B$. We'll prove that $B \subseteq A$; take an arbitrary $y \in B$ and consider $S = \{x, y\}$. Then $S \subseteq A \cup B$ but $S \not\subseteq B$ (since $x \in S$ but $x \notin B$). That is, $S \in \mathcal{P}(A \cup B)$ and $S \notin \mathcal{P}(B)$. By our assumption about $\mathcal{P}(A \cup B)$ we must have $S \in \mathcal{P}(A)$. We've found that $\{x, y\} \subseteq A$, and in particular $y \in A$. This shows that $B \subseteq A$. $\square$

3. Suppose that $f : A \to B$ is an injective function and $S \subseteq A$. Prove that

$$f(A - S) = f(A) - f(S).$$

Proof. Suppose $y \in f(A) - f(S)$ and find $x \in A$ so that $f(x) = y$. If $x \in S$ then we would have $y \in f(S)$, contradicting our choice of y. So $x \in A - S$ and $y \in f(A - S)$. This shows $f(A - S) \subseteq f(A) - f(S)$.

Now suppose $y \in f(A - S)$ and find $x \in A - S$ so that $f(x) = y$. Since $x \in A$ we have $y \in f(A)$. Assume to the contrary that y is an element of $f(S)$ as well. Then there is $z \in S$ so that $f(z) = y$. But then $f(z) = f(x)$, and injectivity of f implies that $x = z$. In particular, $x \in S$, contradicting the fact that $x \in A - S$. So $y \notin f(S)$ and we deduce $y \in f(A) - f(S)$. This shows that $f(A) - f(S) \subseteq f(A - S)$. Altogether we conclude the two sets are equal, as desired. $\square$