Math 8, Summer 2012
Exam 2 Solutions

Short Answer

1. Let S be a set with a relation R. Precisely define what it means for R to be reflexive.

   For all $x \in S$ we have xRx.

2. Evaluate the sum $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$.

   $2^5 = 32$

3. Dirichlet stuffs 316 pigeons into 21 holes. What's the largest number of pigeons we are guaranteed to see in a single hole?

   $15 \cdot 21 = 315 < 316$. Hence we cannot have each hole containing at most 15 pigeons; there must be a hole with at least 16 pigeons.

4. A trichotomous relation $\simeq$ on a set S is one with the following property:

   For all $x, y \in S$, exactly one of the following is true: $x \simeq y$, $y \simeq x$, or $x = y$.

   Give an example of such a relation.

   The best–known example is inequality, $<$, on $\mathbb{R}$.

5. Given an equivalence relation $\sim$ on a set S and $x \in S$, define the equivalence class $[x]$.

   $[x] = \{y \in S : x \sim y\}$

6. Alice wants to prove $P(n)$ for all integers $n \geq 3$. She proves $\forall n \geq 3$, $P(n) \Rightarrow P(n+2)$. Which base cases are needed to complete the induction?

   She needs two base cases, so she should prove $P(3)$ and $P(4)$.

7. Which of these is an equivalence relation on $\mathbb{R}$?

   (a) $x \sim y \iff x - y \in \mathbb{Q}$
   (b) $x \sim y \iff x \leq y$
   (c) $x \sim y \iff x + y \in \mathbb{Z}$
   (d) $x \sim y \iff \sin x = \cos y$
   (e) None of the above

   (a) is the answer. (In brief, $x - x = 0 \in \mathbb{Q}$ for any x, $x - y \in \mathbb{Q}$ implies $y - x = -(x - y) \in \mathbb{Q}$, and whenever $x - y, y - z \in \mathbb{Q}$ we have $x - z = (x - y) + (y - z) \in \mathbb{Q}$)

8. Give a precise statement of the well–ordering principle.

   Any nonempty subset of $\mathbb{N}$ contains a smallest element.

Problems

1. Prove for all integers $n \geq 0$

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

Proof. We proceed by induction on n. When $n = 0$ we have

$$\int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1 = 0!$$

and the claim holds for $n = 0$. Now assume the result holds for some $n \in \mathbb{N}$. It follows that

$$\int_0^\infty x^{n+1} e^{-x}\,dx = -x^{n+1}e^{-x}\Big|_0^\infty + \int_0^\infty (n+1)x^n e^{-x}\,dx = (0-0) + (n+1)\cdot n! = (n+1)!,$$

so the result holds for all $n \geq 0$. □

2. On a spherical surface, define a closed hemisphere to be a hemisphere which includes its edge as part of the set. Given 5 arbitrary points on the surface of a sphere prove there is a closed hemisphere containing 4 of them.

   Partial Solution. If we cut the sphere in half arbitrarily, we have two hemispheres and 5 points. By the pigeonhole principle, some 3 points are on the same hemisphere. □

   Full Solution. If we cut the sphere in half through 2 of the 5 points (that is, make sure some 2 points are on the equator), we have 2 hemispheres and 3 other points. By the pigeonhole principle, some 2 are on the same hemisphere. Together with the 2 on the equator, we have 4 on a closed hemisphere. □

3. Fix a positive integer n. Given an integer $k \geq 0$ define $d(k)$ to be the number of permutations of k objects leaving no objects in their starting positions. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} d(k) = n!$$

Proof. Consider permuting n objects in a row. There are $n!$ ways to do this. Alternatively, we could choose some number k of the objects that will move in $\binom{n}{k}$ ways and "derange" them—that is, permute without fixed points—in $d(k)$ ways. The other unchosen objects stay in their original positions. Summing over all k gives all ways of rearranging the objects, so the result follows. □