Math 118B, Winter 2012
Final Solutions

March 22, 2012

Name____

1. Prove that the function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is infinitely differentiable on $(1, \infty)$.

Proof. Define the functions

$$f_N(s) = \sum_{n=1}^{N} \frac{1}{n^s}$$

and notice that the derivatives satisfy

$$f_N^{(m)}(s) = \sum_{n=1}^{N} \frac{(-\ln n)^m}{n^s}.$$

We'll prove that ζ is smooth on any set $[a, \infty)$, where $a > 1$. Recall the theorem: if f_n converges pointwise and f_n' converges uniformly, then the limit of the derivatives is the derivative of the limit (part of the conclusion is that the pointwise limit of the f_n is differentiable). On $[a, \infty)$ we have

$$\left| \frac{(-\ln n)^m}{n^s} \right| \leq \frac{\ln^m n}{n^a},$$

and $\sum_n (\ln^m n)/n^a$ converges for any m. Therefore $(f_N^{(m)})$ converges uniformly on $[a, \infty)$ for each m by the Weierstrass M-test, implying that $\zeta^{(m)}(s)$ exists on $[a, \infty)$ for each $a > 1$ and $m \in \mathbb{N}$. So ζ is infinitely differentiable on $(1, \infty)$. $\square$

2. Using only the definitions, prove that a space with the property that every sequence admits a convergent subsequence is complete.

Proof. Take a Cauchy sequence (x_n) in the space, which then admits a convergent subsequence (x_{n_k}) with some limit L. Given $\epsilon > 0$ find N so that $d(x_n, x_m) < \epsilon/2$ for all $n, m \geq N$. Find a point x_{n_k} in the subsequence with $n_k \geq N$ and $d(x_{n_k}, L) < \epsilon/2$. Then for all $n \geq N$,

$$d(x_n, L) \leq d(x_n, x_{n_k}) + d(x_{n_k}, L) < \epsilon.$$

□

3. Determine whether $\int_1^\infty \sin^2\left(\frac{\ln x}{x}\right)\,dx$ converges.

Proof. Since $\ln x/x \geq 0$ and $\sin^2(\ln x/x) \geq 0$ on $[1,\infty)$ we have

$$\int_1^\infty \sin^2\left(\frac{\ln x}{x}\right)\,dx \leq \int_1^\infty \left(\frac{\ln x}{x}\right)^2\,dx,$$

and it suffices to show that the latter integral converges. Recall that $\ln x < x$ for all $x \geq 1$; thus $\ln x^{1/3} < x^{1/3}$, or $\ln x < 3x^{1/3}$. Therefore

$$\int_1^\infty \sin^2\left(\frac{\ln x}{x}\right)\,dx \leq \int_1^\infty \frac{9}{x^{4/3}}\,dx < \infty,$$

so the original integral converges. □

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic. Furthermore for each nonnegative integer n,

$$\int_0^{2\pi} e^{in\theta} f(\theta)\, d\theta = 0.$$

Prove that $f \equiv 0$.

Proof. Taking complex conjugates, we have the orthogonality condition for all integers n, not just positive ones. Linearity of the integral implies that

$$\int_0^{2\pi} p(\theta) f(\theta)\, d\theta = 0$$

for any trigonometric polynomial p. The trig polynomials are dense in the space of 2π-periodic continuous functions (in the uniform metric), so we can find $p_n \to f$ uniformly. On a bounded interval, uniform convergence allows the interchange of limit and integral, so

$$0 = \lim_{n\to\infty} \int_0^{2\pi} p_n(\theta) f(\theta)\, d\theta = \int_0^{2\pi} f^2(\theta)\, d\theta.$$

But f^2 is continuous, nonnegative and integrates to zero. Hence $f \equiv 0$. □

5. Prove there exists a unique continuous function $f : [0,1] \to \mathbb{R}$ so that

$$f(x) - \int_0^1 f(x-y)e^{-y}\,dy = \arctan(x).$$

Proof. Define the linear operator $T : C[0,1] \to C[0,1]$ by

$$Tf(x) = \arctan(x) + \int_0^1 f(x-y)e^{-y}\,dy.$$

Given $f, g \in C[0,1]$ we have that

$$\|Tf - Tg\|_\infty = \left\| \int_0^1 (f(x-y) - g(x-y))e^{-y}\,dy \right\|_\infty \leq \|f-g\|_\infty \int_0^1 e^{-y}\,dy.$$

Thus $\|Tf - Tg\|_\infty \leq e^{-1}\|f-g\|_\infty$, and T is a contraction. Since $C[0,1]$ is a complete metric space, the contraction mapping theorem implies that T has a unique fixed point. $\square$

6. Give an example of a subset $\mathcal{F} \subset C[0,1]$ which is uniformly bounded but not precompact (in the uniform metric).

Proof. Let $\mathcal{F} = \{x^n : n \in \mathbb{N}\}$. This family is uniformly bounded by 1, but any subsequence (x^{n_k}) converges pointwise to the discontinuous function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

so the convergence cannot be uniform. That is, no sequence in $\mathcal{F}$ has a uniformly convergent subsequence to a function in $C[0,1]$. $\square$

7. Fix $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Find the power series expansion at $x = 0$ of the function f which solves

$$\begin{cases} (1-x)f'(x) + \alpha f(x) = 0 \\ f(0) = 1 \end{cases}$$

and compute its radius of convergence.

Proof. Let $f(x) = 1 + \sum_{n\geq 1} a_n x^n$. Substitute into the differential equation and perform tedious index chasing to find

$$0 = (\alpha + a_1) + \sum_{n=1}^{\infty} \left[(n+1)a_{n+1} - (n+\alpha)a_n\right] x^n$$

for all x in a neighborhood of $x = 0$. Thus each coefficient vanishes and we find

$$\begin{cases} a_1 = -\alpha \\ a_{n+1} = a_n(n+\alpha)/(n+1) \qquad \text{for all } n \geq 1 \end{cases}$$

This recurrence lends itself to the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+\alpha}{n+1}\right| = 1,$$

so the radius of convergence is 1. The first few terms of the series are

$$f(x) = 1 - \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 - \frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3 + \cdots .$$

By the way, $f(x) = (1-x)^\alpha$. □

8. Suppose that $f : [a, b] \to \mathbb{R}$ is Riemann integrable and that $F' = f$ for some function F. Prove that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Given a partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ rewrite $F(b) - F(a)$ as a telescoping series and use the mean-value theorem:

$$F(b) - F(a) = \sum_{k=1}^{n} F(x_k) - F(x_{k-1}) = \sum_{k=1}^{n} F'(c_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}),$$

where $c_k \in (x_{k-1}, x_k)$. The equation

$$F(b) - F(a) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

holds for any partition, so taking the mesh to 0 gives the result. □

9. Prove Dini's theorem:

Theorem. *Let X be a compact metric space and (f_n) be a sequence of real-valued continuous functions which converge pointwise to a continuous function f. If the functions satisfy $f_1 \geq f_2 \geq f_3 \geq \cdots$, then the convergence is uniform.*

Proof. Let $\epsilon > 0$. Define the sets $U_n = \{x \in X : f_n(x) - f(x) < \epsilon\}$. These sets are open since each function $f_n - f$ is continuous. The monotonicity of the sequence (f_n) implies that $U_1 \subseteq U_2 \subseteq \cdots$; the pointwise convergence of (f_n) implies that each point in the space lies in some U_n. That is, we have an open cover of the compact space X. There exists a finite subcover $U_1, \ldots, U_N$, but $X = U_1 \cup \cdots \cup U_N = U_N$ due to the nested condition. Furthermore, for any $n \geq N$ we have $X = U_N \subseteq U_n \subseteq X$, so that $U_n = X$. In other words, $f_n(x) - f(x) < \epsilon$ for all $x \in X$ and $n \geq N$, as desired. $\square$

10. Define Lip$[0, 1] \subset C[0, 1]$ as the set of functions f for which there is a constant K such that for all $x, y \in [0, 1]$

$$|f(x) - f(y)| \leq K|x - y|.$$

Suppose that a sequence of functions (f_n) in Lip$[0, 1]$ share the same Lipschitz constant K. Prove that if the sequence converges pointwise, then it converges uniformly.

Proof. If a sequence (f_n) in Lip$[0, 1]$ has a single Lipschitz constant K, then clearly the sequence is equicontinuous. Assuming the sequence converges pointwise, the sequence of numbers $(f_n(0))$ is bounded by some $M > 0$. Thus for any $x \in [0, 1]$ we have

$$|f_n(x)| \leq |f_n(0)| + |f_n(x) - f_n(0)| \leq M + K|x| \leq K + M,$$

so the sequence is uniformly bounded. By Arzelá-Ascoli, the sequence is precompact in the uniform metric.

The sequence f_n converges pointwise to a limit function f, so any uniformly convergent subsequence must have limit f. If the entire sequence did not converge uniformly, there would be $\epsilon > 0$ and a subsequence (f_{n_k}) so that

$$\|f_{n_k} - f\|_\infty \geq \epsilon$$

for all k. But this sequence could not have a subsequence converging uniformly to f, which is a contradiction.

□