# The Fubini Principle in Discrete Math

## Introduction: Double Summation

Consider the problem of summing a collection of numbers that have been doubly indexed:

$$\sum_{i,j} a_{i,j}.$$

For example if $i$ and $j$ both take values among $\{1, 2, 3\}$, the aforementioned sum becomes

$$\sum_{i,j} a_{i,j} = a_{1,1} + a_{1,2} + a_{1,3} + a_{2,1} + a_{2,2} + a_{2,3} + a_{3,1} + a_{3,2} + a_{3,3}.$$

We'd like to find simple ways of organizing and computing sums of this type. First, notice in our example:

$$\sum_{i=1}^{3} \left( \sum_{j=1}^{3} a_{i,j} \right) = (a_{1,1} + a_{1,2} + a_{1,3}) + (a_{2,1} + a_{2,2} + a_{2,3}) + (a_{3,1} + a_{3,2} + a_{3,3}) = \sum_{i,j} a_{i,j}.$$

Similarly, $\sum_j \sum_i a_{i,j}$ gives the same result. This simple observation for finite sums comes from the commutativity and associativity of addition, but it has far reaching consequences. We call this the Fubini principle. From now on we drop the comma in doubly indexed sequences, writing $a_{i,j}$ instead as $a_{ij}$.

**Theorem** (Fubini's Principle)**.** *Given a finite sum indexed by $i$ and $j$ we have*

$$\sum_{i,j} a_{ij} = \sum_i \left( \sum_j a_{ij} \right) = \sum_j \left( \sum_i a_{ij} \right).$$

We omit the proof, which is merely uses induction on the size of the sum and basic properties of addition. Here is a simple and well–known application, sometimes called the handshake lemma.

**Theorem.** *In a room of people, some pairs shake hands and some don't. No two people shake hands more than once and nobody shakes her own hand. Given a person $p$, let $n(p)$ denote the number of hands $p$ shook. If the total number of handshakes is $H$, then*

$$\sum_p n(p) = 2H.$$

*Proof.* Given a person $p$ and handshake $h$ let $\mathbb{1}_{ph}$ be 1 if person $p$ participated in handshake $h$ and 0 otherwise. If we fix a person $p$ and sum $\sum_h \mathbb{1}_{ph}$, we obtain the number of hands $p$ shook. That is,

$$\sum_h \mathbb{1}_{ph} = n(p).$$

If we fix a handshake $h$ and sum $\sum_p \mathbb{1}_{ph}$ we obtain 2, the number of people involved in any single handshake. This gives

$$\sum_p n(p) = \sum_p \sum_h \mathbb{1}_{ph} = \sum_h \sum_p \mathbb{1}_{ph} = \sum_h 2 = 2H,$$

as desired. □

As a corollary, if 5 people in a room each claim to have shaken 3 hands then someone is lying—the number 15 is not even.

## Switching Sums, Changing Indices

In the previous section we only examined sums whose indices varied independently. There are many situations where the indices are constrained by each other:

1. Sum over $i, j \in \{1, 2, 3, 4\}$ but only include terms with $i < j$.
2. Sum over all subsets $A \subseteq S$ and elements $x \in A$.
3. Sum over all people $p$ in a room and each friend of $p$.
4. Etc.

The sum–switching principle is no different in this case; rather, we need to learn how to describe the indices in multiple ways.

*Example.* Sum the series

$$\sum_{i=0}^{n} \sum_{j=i}^{n} \binom{j}{i}.$$

We understand sums of the form $\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \cdots$ better than the sum written here. It would thus be advantageous to reverse the order and sum over $i$ first. However, blind manipulation yields

$$\sum_{j=i}^{n} \sum_{i=0}^{n} \binom{j}{i}, \qquad \text{(nonsense!)}$$

which is meaningless! The "answer" to our problem would depend upon a dummy variable $i$, which makes no sense. Instead, we revert to the double sum:

$$\sum_{i=0}^{n} \sum_{j=i}^{n} \binom{j}{i} = \sum_{0 \le i \le j \le n} \binom{j}{i},$$

and try to describe the double sum in the desired order. We want $i$ on the 'inside' sum, so let's decide the bounds of $j$ first. Ignoring $i$ temporarily, we see that $j$ varies from 0 to $n$. At a fixed $j$ between 0 and $n$ (which is how the inner summation sees $j$), the variable $i$ can be as small as 0 and as large as $j$. That is,

$$\sum_{i=0}^{n} \sum_{j=i}^{n} \binom{j}{i} = \sum_{j=0}^{n} \sum_{i=0}^{j} \binom{j}{i}.$$

We know the inner sum is merely $2^j$, so the rest is easy. We compute

$$\sum_{i=0}^{n} \sum_{j=i}^{n} \binom{j}{i} = \sum_{j=0}^{n} 2^j = 2^{n+1} - 1,$$

as desired. □

Here's a different problem.

*Example.* Let $S$ be a set of $n$ elements. Evaluate the sum

$$\alpha = \sum_{A, B \subseteq S} |A \cup B|.$$

We rewrite the sum as a double sum and switch the order. The outer sum becomes a sum over all elements $x \in S$. The inner sum must involve sets for which $x \in A \cup B$.

$$\alpha = \sum_{A, B \subseteq S} \sum_{x \in A \cup B} 1 = \sum_{x \in S} \sum_{\substack{A, B \subseteq S \\ x \in A \cup B}} 1.$$

This inner sum is the number of ways that $x$ can appear in $A \cup B$ when we take arbitrary $A, B \subseteq S$. When choosing these subsets from $S$, there are 4 choices for each element $y \in S$: either $y \in A \cap B$, $y \in A - B$, $y \in B - A$, or $y \notin A \cup B$. The exception is the element $x$, for which there are only 3 choices—we can't have $x \notin A \cup B$. Thus the inner sum is $3 \cdot 4^{n-1}$, giving us

$$\alpha = \sum_{x \in S} \sum_{\substack{A,B \subseteq S \\ x \in A \cup B}} 1 = \sum_{x \in S} 3 \cdot 4^{n-1} = 3n4^{n-1},$$

and we're done. □

## Similarities to Integration

Before we turn to applications, let's briefly examine the analgous Fubini theorem in integration. This section is only meant to help the reader familiar with integration over regions in the plane; a reader with no background in multivariable calculus can skip to the next section.

Consider a scalar function $f : \mathbb{R}^2 \to \mathbb{R}$ and a bounded region $R \subseteq \mathbb{R}^2$. We wish to study a weighted sum of areas:

$$\sum_{i=1}^{N} f(p_i)\, \Delta A_i,$$

where $R$ has been partitioned into $N$ disjoint regions with areas $\Delta A_1, \Delta A_2, \ldots$ and points $p_1, p_2, \ldots$ have been chosen as representatives from each region. If we take $N \to \infty$ so that all $\Delta A_i \to 0$ in a 'nice' way, these weighted sums converge to the area integral

$$\int_R f\, dA,$$

assuming $f$ is sufficiently well-behaved. To evaluate this integral we turn to two theorems of Fubini and Tonelli, which we combine here into one theorem.

**Theorem** (Fubini)**.** *Let $R \subseteq \mathbb{R}^2$ and suppose that $f : R \to \mathbb{R}$ is a measurable[1] function such that either $f \geq 0$ throughout $R$ or*

$$\int_R |f|\, dA < \infty.$$

*Then it follows that*

$$\int_R f\, dA$$

*exists and can be evaluated as an iterated integral in either order.*

Essentially any function we'd want to integrate will satisfy one of these conditions, so we typically take this theorem for granted. The most common example of this theorem in action is integration over triangles.

*Example.* Let's integrate $f(x, y) = ye^{x^2}$ over the triangular region $R$ with vertices $(0, 0), (1, 1), (1, 0)$. The region can be described as $0 \leq y \leq 1$ and $y \leq x \leq 1$ leading to the impossible integral

$$\int_R ye^{x^2}\, dA = \int_0^1 \left( \int_y^1 ye^{x^2}\, dx \right) dy.$$

If instead we describe the set as $0 \leq x \leq 1$ and $0 \leq y \leq x$ we obtain the easy integral

$$\int_R ye^{x^2}\, dA = \int_0^1 \left( \int_0^x ye^{x^2}\, dy \right) dx = \int_0^1 xe^{x^2}\, dx = \frac{e-1}{2}.$$

Fubini's theorem makes all of these computations possible. □

Fubini's theorem is not limited to integrals. If we wish to swap infinite sums, we need some sort of justification. Fortunately Fubini's theorem applies to infinite sums as well—in a sufficiently abstract setting, sums and integrals are equivalent and can be treated with the same theorems and techniques.

[1] Technical term; disregard. In practice all functions are measurable.

**Theorem** (Fubini for sums). *Suppose that $a_{jk}$ is a doubly indexed infinite sequence of real (or complex) numbers. Suppose either $a_{jk} \geq 0$ for all indices $j, k$ or*

$$\sum_{j,k} |a_{jk}| < \infty.$$

*Then $\sum a_{jk}$ exists and*

$$\sum_{j,k} a_{jk} = \sum_j \sum_k a_{jk} = \sum_k \sum_j a_{jk}.$$

As a final example in this section, let's look at a situation wherein sum swapping is not allowed.

*Example.* Let's define $a_{jk}$ for positive integers $j, k$. Whenever $j = k$ set $a_{jk} = 1$ and whenever $j = k + 1$ let $a_{jk} = -1$. Otherwise, let $a_{jk} = 0$. Then for any $k$ we have $\sum_j a_{jk} = 0$ and it follows that

$$\sum_k \sum_j a_{jk} = 0.$$

When $j \geq 2$ we have $\sum_k a_{jk} = 0$, but when $j = 1$ we have $\sum_k a_{jk} = 1$. It follows that

$$\sum_j \sum_k a_{jk} = 1 \neq \sum_k \sum_j a_{jk}.$$

This phenomenon can only happen for infinite sums with both positive and negative terms, but it's worth keeping in mind. □

## Applications

We can compute binomial sums using Fubini and simpler identities we already know.

*Example.* Sum the series

$$S = \sum_{j,k} \binom{m}{j} \binom{r}{k-j} \binom{k}{r}.$$

Writing as an iterated sum with $j$ on the inside gives

$$S = \sum_k \binom{k}{r} \left( \sum_j \binom{m}{j} \binom{r}{k-j} \right).$$

The sum in parentheses is given by the so–called Vandermonde identity:

$$S = \sum_k \binom{k}{r} \binom{m+r}{k}.$$

The product of binomials here can be rewritten so that one of them does not involve the index $k$:

$$S = \sum_k \binom{m}{k} \binom{m+r}{r} = \binom{m+r}{r} \sum_k \binom{m}{k}.$$

This last sum is simply $2^m$, so the final answer is $S = \binom{m+r}{r} 2^m$. □

Here's a harder example. This problem uses permutations as bijective functions from a set to itself.

*Example.* Let $S = \{1, 2, \ldots, n\}$. Given a permutation $\pi : S \to S$ define

$$f(\pi) = \sum_{k=1}^{n} |k - \pi(k)|.$$

Find the average value $A$ of $f$ over all $n!$ permutations of $S$. That is, we wish to compute

$$A = \frac{1}{n!} \sum_\pi \sum_{k=1}^{n} |k - \pi(k)|.$$

Reversing the sum gives

$$A = \frac{1}{n!} \sum_{k=1}^{n} \sum_{\pi} |k - \pi(k)|.$$

To understand this inner sum, consider $k$ fixed. For each $j \in \{1, 2, \ldots, n\}$ note that $\pi(k) = j$ for $(n-1)!$ permutations. That is,

$$A = \frac{1}{n!} \sum_{k=1}^{n} \sum_{j=1}^{n} (n-1)!|k - j| = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n} |k - j|.$$

To evaluate this sum, notice that the $j = k$ terms are 0 and that

$$A = \frac{1}{n} \sum_{1 \le j \le k \le n} |k - j| + \frac{1}{n} \sum_{1 \le k \le j \le n} |k - j| = \frac{2}{n} \sum_{1 \le j \le k \le n} (k - j) = \frac{2}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} (k - j).$$

This sum is easy to evaluate:

$$\begin{aligned} A &= \frac{2}{n} \sum_{k=1}^{n} \left( k^2 - \frac{k(k+1)}{2} \right) = \frac{1}{n} \sum_{k=1}^{n} (k^2 - k) \\ &= \frac{1}{n} \left( \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right) \\ &= \frac{(n+1)(2n+1) - 3(n+1)}{6} \\ &= \frac{n^2 - 1}{3}, \end{aligned}$$

and we're done. □