Math 8 Homework 1 Solutions

1 Logical Statements

(a) A set S is not compact iff either S isn't closed or S isn't bounded.

(b) We can write uniform continuity as

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ s.t } \forall x, y \in \mathbb{R}, \; |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon.$$

The negation is

There is $\epsilon > 0$ so that for any $\delta > 0$ there are $x, y \in \mathbb{R}$ so that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

(c) (i) This is true, since the size of G would be a multiple of 14, hence even.

(ii) Actually, H could have size 1, which is not even.

(iii) This need not follow. This is a well–known fact about the so–called alternating group A_4.

2 Direct and Contradiction Proof

(a) *Proof.* ($\Rightarrow$) Suppose that n is even. Then there is $k \in \mathbb{Z}$ so that $n = 2k$. But then $n^2 = 2 \cdot (2k^2)$, which is even.

($\Leftarrow$) Suppose that n is not even. Then there is $k \in \mathbb{Z}$ so that $n = 2k + 1$. Then $n^2 = 2 \cdot (2k^2 + 2k) + 1$, which is not even. □

(b) *Proof.* Assume to the contrary that $\sqrt{2}$ is rational. Then we can write $\sqrt{2} = m/n$, where not both m, n are even. It follows that $m^2 = 2n^2$. Since m^2 is even, m is even as well (by the previous problem). We can write $m = 2k$ for some integer k. This implies $n^2 = 2k^2$, which implies n is even. This however contradicts our assumption that not both m and n should be even. Hence $\sqrt{2}$ is irrational. □

(c) This is sort of a trick question. Notice that both $\sqrt{2}$ and $-\sqrt{2}$ are irrational; their sum (zero) is rational.

(d) *Proof.* Let $x = \sqrt{2}^{\sqrt{2}}$. It's not clear whether x is rational; if it is we're done. If however x is irrational, then notice that $x^{\sqrt{2}} = 2$, a rational number. So either x or $x^{\sqrt{2}}$ satisfies the required condition. □

(e) The proof is invalid because it assumes its conclusion to be true. Note that the same sort of reasoning can show that $-1 = 1$ (proof: square both sides). Rather, we can can proceed as follows:

Proof. Assume to the contrary that $\sqrt{6} + \sqrt{2} \geq 4$. Since all quantities are positive, we can square both sides to obtain $8 + 2\sqrt{12} \geq 16$. Then we have $\sqrt{12} \geq 4$. Again we can square to find $12 \geq 16$. This contradiction proves the result. □

3 Quantifiers

(a) *Proof.* Let $f(x) = (1 - x)\cos x - \sin x$. Notice that $f(0) = 1 > 0$ and $f(1) = -\sin 1 < 0$. By the intermediate value theorem f has a root in $(0, 1)$, as desired. □

(b) *Proof.* Define $g(x) = f(x+1) - f(x)$. If either $g(0)$ or $g(1)$ is zero, we're done. Otherwise, $g(1) = f(1) - f(0) = f(1) - f(2) = -g(0)$. Whether $g(0)$ is positive or negative, then $g(1)$ is the opposite sign; by the intermediate value theorem, g has a root in $(0, 1)$, as desired. □

(c) *Proof.* Let $\epsilon > 0$. Find an integer N so that $N > 1/\epsilon$. Given any integers $n > m \geq N$ we have

$$\frac{1}{m} - \frac{1}{n} < \frac{1}{m} \leq \frac{1}{N} < \epsilon.$$

□

(d) When $n = 41$ we have $n^2 + n + 41 = 41 \cdot 43$, which is not prime.

(e) *Proof.* Assume to the contrary that $f'(x) \geq 1 + f(x)^2$ for all $x \in (0, 4)$. Then we can write

$$\frac{f'(x)}{1 + f(x)^2} \geq 1.$$

Integrating from 0 to 4 preserves the inequality:

$$\arctan f(4) - \arctan f(0) = \int_0^4 \frac{f'(x)}{1 + f(x)^2}\, dx \geq \int_0^4 dx = 4.$$

No matter what values $f(0)$ and $f(4)$ are, two values of arctan can never differ by more than π. We conclude that $\pi > 4$, a contradiction. ☐