# Math 8 Homework 2 Solutions

## 1 Set Basics

(a) False. If $A = \varnothing$ and $B = \{1\}$ is nonempty, then $A \cup B = \{1\}$ while $A \cap B = \varnothing$.

(b) True. The statement $\varnothing \in \mathcal{P}(A)$ is equivalent to the statement $\varnothing \subseteq A$; that is, whenever $x \in \varnothing$ we also have $x \in A$. This "if, then" statement is vacuously true since $x \in \varnothing$ is never true.

(c) True. Note that $\{\varnothing\} \in \mathcal{P}(\{\varnothing, \{\varnothing\}\})$ is equivalent to $\{\varnothing\} \subseteq \{\varnothing, \{\varnothing\}\}$. This occurs if and only if $\varnothing \in \{\varnothing, \{\varnothing\}\}$, which is true.

(d) False. Consider $A = \{1\}$ and $B = \{2\}$. Then $\{1, 2\} \subseteq A \cup B$, although $\{1, 2\}$ is a subset of neither $A$ nor $B$. This means $\{1, 2\} \in \mathcal{P}(A \cup B)$, whereas $\{1, 2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

(e) False. No matter what $A$ and $B$ are, $\varnothing \in \mathcal{P}(A \times B)$. On the other hand, $\varnothing \notin \mathcal{P}(A) \times \mathcal{P}(B)$ since the product of sets contains only ordered pairs.

## 2 Set Proofs

(a) *Proof.* ($\Leftarrow$) Suppose that $A \subseteq \varnothing$. Since $\varnothing \subseteq A$ no matter what $A$ is, this shows that $A = \varnothing$.
($\Rightarrow$) Suppose that $A = \varnothing$. This immediately implies that $A \subseteq \varnothing$. $\square$

(b) *Proof.* ($\Leftarrow$) Suppose that $A \cup B = A$. Given an arbitrary $x \in B$, we see that $x \in A \cup B = A$. Hence $B \subseteq A$.
($\Rightarrow$) Suppose that $B \subseteq A$. Given $x \in A \cup B$, either $x \in A$ or $x \in B$; however $B \subseteq A$ implies that in either case $x \in A$. Therefore $A \cup B \subseteq A$. On the other hand, $A \subseteq A \cup B$ is generally true. We conclude that $A = A \cup B$. $\square$

(c) *Proof.* Suppose that $A \neq \varnothing$ and $A \times B = \varnothing$. Assume for sake of contradiction that $B \neq \varnothing$. We can find $x \in A$ and $y \in B$, so that $(x, y) \in A \times B$, contradicting the fact that $A \times B$ is empty. Hence $B = \varnothing$. $\square$

(d) *Proof.* Suppose that $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$ and that $A \cap B$ is nonempty. Fix $x \in A \cap B$ and assume we can find $y \in A$ such that $y \notin B$. Since $x$ and $y$ are both elements of $A$, we have $\{x, y\} \subseteq A$. On the other hand, since $y \notin B$ we see that $\{x, y\} \not\subseteq B$. Therefore $\{x, y\} \in \mathcal{P}(A) - \mathcal{P}(B)$. We deduce that $\{x, y\} \in \mathcal{P}(A - B)$; that is, $\{x, y\} \subseteq A - B$. In particular $x \in A - B$, which implies that $x \notin B$. This contradicts the fact that $x \in A \cap B$, so our assumption that $y$ exists was false. We conclude that $A \subseteq B$ as desired. $\square$

(e) *Proof.* Assume that $A \cup B \subseteq C \cup D$, $A \cap B = \varnothing$, and $C \subseteq A$. Let $x \in B$ and note that $x \notin A$, lest $A \cap B$ be nonempty. It follows that $x \notin C$. We know that $x$ is an element of $A \cup B$, hence of $C \cup D$ as well. Since $x \notin C$, we conclude that $x$ must be an element of $D$. That is, $B \subseteq D$. $\square$

(f) *Proof.* Let $x \in A \Delta \varnothing$. Then either $x \in A - \varnothing$ or $x \in \varnothing - A$. If $x$ were an element of $\varnothing - A$, then $x$ would be an element of the empty set. This cannot be, so $x \in A - \varnothing$. In particular, $x \in A$ and we deduce that $A \Delta \varnothing \subseteq A$. Now consider an arbitrary $y \in A$. Since $y \notin \varnothing$, we have $y \in A - \varnothing$. We conclude that $y \in A \Delta \varnothing$, hence $A \subseteq A \Delta \varnothing$. It follows that $A = A \Delta \varnothing$. $\square$

(g) *Proof.* ($\Leftarrow$) Suppose that $A \Delta B = \varnothing$. Assume we can find an $x$ in $A$ but not in $B$. Then $x \in A - B$, hence $x \in A \Delta B$. This contradicts the fact that $A \Delta B$ is empty, so no such $x$ exists. That is, $A \subseteq B$. The same argument shows that $B \subseteq A$ as well, so $A = B$.
($\Rightarrow$) Suppose that $A = B$. Then any element of $A$ is an element of $B$ and vice versa, so both $A - B$ and $B - A$ are empty. That is, $A \Delta B = \varnothing$. $\square$

(h) The statement is true. Let $x \in (A \Delta B) \Delta C$. Then either $x \in (A \Delta B) - C$ or $x \in C - (A \Delta B)$. First suppose that $x \in (A \Delta B) - C$; then $x \in A \Delta B$ and $x \notin C$. Moreover, either $x \in A - B$ or $x \in B - A$. If $x \in A - B$ then $x \in A$ and $x \notin B$. Since $x \notin C$ as well, we have $x \notin B \Delta C$ and finally $x \in A \Delta (B \Delta C)$. If instead $x \in B - A$ then from $x \notin C$ we have $x \in B - C$. Thus $x \in B \Delta C$; since $x \notin A$, we see $x \in A \Delta (B \Delta C)$.

Now, returning to the beginning, suppose that $x \in C - (A \Delta B)$ instead. Then $x \in C$ and $x \notin A \Delta B$. If $x \in A$ then $x \in B$ as well, lest $x \in A \Delta B$. Since $x$ is an element of both $B$ and $C$, we have $x \notin B \Delta C$. Hence $x \in A \Delta (B \Delta C)$. On the other hand if $x \notin A$, then $x \notin B$ as well—otherwise $x \in A \Delta B$ would follow. Since $x$

is an element of $C$ but not $B$, we have $x \in B\Delta C$. Together with the fact that $x \notin A$, we find $x \in A\Delta(B\Delta C)$. Finally we conclude that $(A\Delta B)\Delta C \subseteq A\Delta(B\Delta C)$.

The same argument proves the opposite inclusion, so $(A\Delta B)\Delta C = A\Delta(B\Delta C)$, as desired. Whew.

# 3 Indexed Collections

(a) The statement is not generally true. Suppose that $B$ is any nonempty set and that some $A_i = B$ while another $A_j = \varnothing$. Then the intersection of the sets $(A_i)$ is empty and we have

$$B - \left(\bigcap_{i\in I} A_i\right) = B - \varnothing = B.$$

On the other hand one of the sets $B - A_i$ is empty as well, so

$$\bigcap_{i\in I}(B - A_i) = \varnothing.$$

As we can see, the two sets are different.

(b) *Proof.* For simplicity we denote $A_0 = \{x \in \mathbb{R} : f(x) > 0\}$ and $A_n = \{x \in \mathbb{R} : f(x) \geq 1/n\}$ for each $n \in \mathbb{N}$. Given $x \in A_0$ we can find $m \in \mathbb{N}$ so that $1/m \leq f(x)$ (we'll accept this as intuitively clear, but one could prove this fact carefully). Thus $x \in A_m$ and moreover $x \in \cup_n A_n$. Conversely if $x$ is an element of the aforementioned union, then for some $m$ we have $x \in A_m$. This implies that $f(x) \geq 1/m$. In particular $f(x) > 0$, so $x \in A_0$. □

(c) *Proof.* Let $x \in I$. Then for some $n \in \mathbb{N}$ we have

$$x \in \bigcap_{m=n}^{\infty} A_m.$$

That is, $x$ is an element of $A_m$ for each $m \geq n$ (we could say "$x$ is in all but finitely many of the sets"). To show that $x \in S$, we need to show that

$$x \in \bigcup_{m=N}^{\infty} A_m. \qquad (\star)$$

for an arbitrary $N \in \mathbb{N}$. Indeed, taking $m$ to be the larger of $N$ and $n$ we see that $x$ is an element of $A_m$ as well as the union in equation $(\star)$ above. As $x$ was arbitrary, $x \in S$ implies $I \subseteq S$. □

(d) Consider the sequence of sets $A_1 = \{1\}, A_2 = \varnothing, A_3 = \{1\}, A_4 = \varnothing, A_5 = \{1\}, \ldots$ and note that $\varnothing$ repeatedly appears, even arbitrarily far down the sequence. That is,

$$\bigcap_{m=n}^{\infty} A_m = \varnothing$$

for any $n$. Hence the set $I$ is empty (since it is a union of empty sets). On the other hand,

$$\bigcup_{m=n}^{\infty} A_m = \{1\}$$

for all $n$. Therefore $S = \{1\}$ and $I \neq S$.