Math 8 Homework 3 Solutions

1 Examples of Functions

(a) (i) $\mathbb{N} \to \mathbb{N}$

(ii) $\mathbb{R} \to \mathbb{R}$

(iii) $\mathcal{P}(\mathbb{R}) \to [0, \infty]$

(iv) $\mathbb{R}^3 \to \mathbb{R}^2$

(v) $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

(vi) $C(\mathbb{R}) \to \mathbb{R}$

(vii) $\mathbb{N} \to \mathbb{R}$

(b) (i) *Proof.* Fix an arbitrary set A. Note that $\varnothing \cap A = \varnothing$; thus we can write

$$\phi(A) = \phi(\varnothing \cup A) = \phi(\varnothing) + \phi(A).$$

Subtracting $\phi(A)$ from this equation leaves us with $\phi(\varnothing) = 0$. $\square$

(ii) *Proof.* Let A, B be arbitrary. Note that $A \cup B = A \cup (B - A)$ and that $A \cap (B - A) = \varnothing$. Thus

$$\phi(A \cup B) = \phi(A) + \phi(B - A). \qquad (\star)$$

Next notice that $B = (B - A) \cup (A \cap B)$ and $(B - A) \cap (A \cap B) = \varnothing$. Thus

$$\phi(B) = \phi(B - A) + \phi(A \cap B).$$

Subtracting this from equation $(\star)$ give the result. $\square$

2 Injectivity and Surjectivity

(a) (i) Let $f : \{1\} \to \{1\}$ be the identity function and $g : \{1\} \to \mathbb{R}$ be arbitrary.

(ii) Let $f : \{1\} \to \mathbb{R}$ be arbitrary and $g : \mathbb{R} \to \mathbb{R}$ be the identity function.

(iii) Let $f : \{1\} \to \mathbb{R}$ be arbitrary and $g : \mathbb{R} \to \{1\}$ map every real number to 1.

(iv) Let $f : \mathbb{R} \to \{1\}$ map every real number to 1 and $g : \{1\} \to \{1\}$ be the identity function.

(v) Let $f : \mathbb{R} \to \mathbb{R}$ be the identity function and $g : \mathbb{R} \to \{1\}$ map every real number to 1.

(vi) Let $f : \{1\} \to \mathbb{R}$ be the inclusion map and $g : \mathbb{R} \to \{1\}$ map every real number to 1.

(b) *Proof.* Let $y \in C$ be arbitrary. By the surjectivity of $g \circ f$ we can find $x \in A$ so that $(g \circ f)(x) = y$. But this implies that g maps $f(x)$ to y. So g is surjective. $\square$

(c) *Proof.* Assume that $x, y \in A$ are such that $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$, but the injectivity of $g \circ f$ implies that $x = y$. So f is injective. $\square$

(d) *Proof.* Suppose there are real numbers $x \neq y$ so that $f(x) = f(y)$. By the mean value theorem we can find a number c between x and y so that

$$0 = \frac{f(x) - f(y)}{x - y} = f'(c),$$

contradicting the assumption that f' is never zero. We conclude that f is injective. $\square$

(e) *Proof.* Assume that f is surjective. Then there are $x, y \in (0, 1)$ so that $f(x) = 0$ and $f(y) = 2$. By the mean value theorem there is a c between x and y so that $f(x) - f(y) = f'(c)(x - y)$. But $|x - y| \leq 1$, so we have

$$2 = |f(x) - f(y)| = |f'(c)| \cdot |x - y| \leq 1 \cdot 1 = 1,$$

a contradiction. So f is not surjective. $\square$

3 Images of Sets

(a) Let $f : A \to B$ be an arbitrary function and $C, D \subseteq A$.

(i) *Proof.* Let $y \in f(C \cup D)$ and find $x \in C \cup D$ so that $f(x) = y$. If $x \in C$ then $y \in f(C)$; otherwise $x \in D$ and $y \in f(D)$. Either way $y \in f(C) \cup f(D)$ so we have that $f(C \cup D) \subseteq f(C) \cup f(D)$.

Now let $z \in f(C) \cup f(D)$. If $z \in f(C)$ we can find $a \in C$ so that $f(a) = z$; in this case $a \in C \cup D$ and $z \in f(C \cup D)$. Otherwise $z \in f(D)$, so we can find $b \in D$ such that $f(b) = z$. In this case $b \in C \cup D$ and $z \in f(C \cup D)$. This covers all cases, so $f(C) \cup f(D) \subseteq f(C \cup D)$. We conclude that the sets are equal. $\square$

(ii) *Proof.* Let $y \in f(C \cap D)$ and find $x \in C \cap D$ so that $f(x) = y$. Since $x \in C$ we have $y \in f(C)$; similarly $y \in f(D)$. Hence $y \in f(C) \cap f(D)$ and we deduce $f(C \cap D) \subseteq f(C) \cap f(D)$. $\square$

(iii) Let $f : \{1, 2\} \to \{3\}$ map both 1 and 2 to 3. If we set $C = \{1\}$ and $D = \{2\}$ then

$$f(C \cap D) = f(\varnothing) = \varnothing \neq \{3\} = f(C) \cap f(D).$$

(b) Let $f : A \to B$ be an arbitrary function and $E, F \subseteq B$.

(i) *Proof.* Let $x \in f^{-1}(E \cup F)$. Then either $f(x) \in E$ or $f(x) \in F$; in the first case $x \in f^{-1}(E)$, while in the second case $x \in f^{-1}(F)$. Either way $x \in f^{-1}(E) \cup f^{-1}(F)$, whence $f^{-1}(E \cup F) \subseteq f^{-1}(E) \cup f^{-1}(F)$.

Now let $y \in f^{-1}(E) \cup f^{-1}(F)$. Then either $f(y) \in E$ or $f(y) \in F$. Either way, $f(y) \in E \cup F$, so we deduce $y \in f^{-1}(E \cup F)$ and $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$. $\square$

(ii) *Proof.* Let $x \in f^{-1}(E \cap F)$, so that $f(x) \in E \cap F$. Then $f(x) \in E$ and $f(x) \in F$; that is, $x \in f^{-1}(E)$ and $x \in f^{-1}(F)$. From this we deduce $f^{-1}(E \cap F) \subseteq f^{-1}(E) \cap f^{-1}(F)$.

Now assume $y \in f^{-1}(E) \cap f^{-1}(F)$. Since $y \in f^{-1}(E)$ we have $f(y) \in E$. Since $y \in f^{-1}(F)$ we have $f(y) \in F$. Hence we have $f(y) \in E \cap F$ and $y \in f^{-1}(E \cap F)$. We conclude that $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$. $\square$