Math 8 Homework 4 Solutions

1 Orderings and Other Relations

(a) The relation $\neq$ on any set is irreflexive.

(b) Suppose a nonempty set A has an antisymmetric equivalence relation $\sim$. If x and y are elements of A so that $x \sim y$, then symmetry implies $y \sim x$ and antisymmetry implies $x = y$. Conversely, reflexivity implies that $x \sim y$ whenever $x = y$; that is, $x \sim y$ if and only if $x = y$. The relation $\sim$ is simply $=$.

(c) *Proof.* Denote the underlying set by A. Given $x \in A$ we have $x \leq x$ and hence $x \geq x$, whence $\geq$ is reflexive. Given $x, y \in A$ so that $x \geq y$ and $y \geq x$ we have $y \leq x$ and $x \leq y$; the antisymmetry of $\leq$ implies $x = y$ and consequentially $\geq$ is antisymmetric. Now assume $x, y, z \in A$ satisfy both $x \geq y$ and $y \geq z$. By definition it follows that $y \leq x$ and $z \leq y$, so the transitivity of $\leq$ gives $z \leq x$. From here we have $x \geq z$ and conclude that $\geq$ is transitive. $\square$

(d) Let $A = \{1, 2\}$ and consider the relation $\subseteq$ on $\mathcal{P}(A)$. Any set $S \in \mathcal{P}(A)$ satisfies $S \subseteq S$, so $\subseteq$ is reflexive. If sets $S, T \in \mathcal{P}(A)$ satisfy $S \subseteq T$ and $T \subseteq S$, then $S = T$ by definition of set equality; hence $\subseteq$ is antisymmetric. Finally, if $S, T, U \in \mathcal{P}(A)$ satisfy both $S \subseteq T$ and $T \subseteq U$ then it follows that $S \subseteq U$; indeed, if $x \in S$ then $x \in T$, whence $x \in U$. We conclude that $\subseteq$ is a partial ordering.

The relation is not a total ordering, however. Note that $\{1\}, \{2\} \in \mathcal{P}(A)$ but neither $\{1\} \subseteq \{2\}$ nor $\{2\} \subseteq \{1\}$ is true.

(e) TYPO ALERT: I should've said $x \leq z$ for all $x \in S \cup \{z\}$ to ensure $\leq$ is reflexive. My bad.

Proof. Let $x \in S \cup \{z\}$. If $x \neq z$ then $x \leq x$ since $\leq$ is reflexive on S. If $x = z$ then $x \leq x$ by our new definition. Either way, $\leq$ is reflexive.

Let $x, y \in S \cup \{z\}$ satisfy both $x \leq y$ and $y \leq x$. If $y = z$ then $z \leq x$ implies $x = z$ as well (since the only thing we stated z to be less than is itself). Similarly if $x = z$, $y = z$ follows. Finally if both $x, y \in S$ then $x = y$ by antisymmetry of $\leq$ on S.

Suppose that $a \leq b$ and $b \leq c$ for some $a, b, c \in S \cup \{z\}$. As above, if $a = z$ then $b = c = z$ as well and $a \leq c$ follows. If $b = z$ then $c = z$ as well and $a \leq c$ follows. If $c = z$ then $a \leq z$ no matter what. Finally if $a, b, c \in S$ then $a \leq c$ by transitivity of $\leq$ on S. We conclude that $\leq$ is transitive. $\square$

(f) For simplicity denote the x–axis by X and the y–axis by Y.

The relation is not transitive; note that $X \perp Y$ and $Y \perp X$, and yet $X \not\perp X$.

The relation is symmetric. Whenever $\ell_1 \perp \ell_2$ we have $\ell_2 \perp \ell_1$.

The relation is not antisymmetric; for example $X \perp Y$ and $Y \perp X$, but $X \neq Y$.

The relation is not reflexive; no line is perpendicular to itself!

2 Equivalence Relations

(a) (i) *Proof.* Let $x \in \mathbb{Z}$. Then $x - x = 7 \cdot 0$, so $x \equiv x$.

Let $x, y \in \mathbb{Z}$ satisfy $x \equiv y$. Find $k \in \mathbb{Z}$ so that $x - y = 7k$. Then $y - x = 7(-k)$, so $y \equiv x$.

Let $x, y, z \in \mathbb{Z}$ satisfy $x \equiv y$ and $y \equiv z$. Find integers k, ℓ so that $x - y = 7k$ and $y - z = 7\ell$. Then $x - z = 7(k + \ell)$, so $x \equiv z$. $\square$

(ii) *Proof.* Let $x \in \mathbb{R}$. Then $x - x = 0 \in \mathbb{Z}$, so $x \simeq x$.

Let $x, y \in \mathbb{R}$ satisfy $x \simeq y$. Find $k \in \mathbb{Z}$ so that $x - y = k$. Then $y - x = -k$, so $y \simeq x$.

Let $x, y, z \in \mathbb{R}$ satisfy $x \simeq y$ and $y \simeq z$. Find integers k, ℓ so that $x - y = k$ and $y - z = \ell$. Then $x - z = k + \ell$, so $x \simeq z$. $\square$

(iii) *Proof.* Let $(x, y) \in [0, 1]^2$. Then $(x, y) \sim (x, y)$ by defintion of $\sim$.

Let $(x, y), (w, z) \in [0, 1]^2$ satisfy $(x, y) \sim (w, z)$. If $(x, y) = (w, z)$ then $(w, z) \sim (x, y)$ follows. Otherwise, $x = w$ and $y + z = 1$. But then $w = x$ and $z + y = 1$, so $(w, z) \sim (x, y)$.

Let $(x, y), (w, z), (a, b) \in [0, 1]^2$ satisfy $(x, y) \sim (w, z)$ and $(w, z) \sim (a, b)$. If any two of the pairs are equal then $(x, y) \sim (a, b)$ follows immediately. Therefore assume that $x = w$, $y + z = 1$, $w = a$ and $z + b = 1$; from this we deduce that $x = a$ and $y + b = 1$, so $(x, y) \sim (a, b)$. □

(iv) *Proof.* Let $(x, y) \in [0, 1]^2$. Then $(x, y) \approx (x, y)$ by defintion of $\approx$.

Let $(x, y), (w, z) \in [0, 1]^2$ satisfy $(x, y) \approx (w, z)$. If $(x, y) = (w, z)$ then $(w, z) \approx (x, y)$ follows. Otherwise, both (x, y) and (w, z) are elements of ∂S, so $(w, z) \approx (x, y)$ follows.

Let $(x, y), (w, z), (a, b) \in [0, 1]^2$ satisfy $(x, y) \approx (w, z)$ and $(w, z) \approx (a, b)$. If any two of the pairs are equal then $(x, y) \approx (a, b)$ follows immediately. Otherwise all three pairs are elements of ∂S, in which case $(x, y) \approx (a, b)$ follows immediately. □

(b) (i) There are 7 equivalence classes—every integer is equivalent to one among $\{0, 1, 2, \ldots, 6\}$. Each equivalence class is corresponds to the remainder upon division by 7.

(ii) We've identified $[0, 1]$ to $[1, 2]$ and $[2, 3]$, etc. In some sense we've collapsed $\mathbb{R}$ to simply $[0, 1]$, but 0 is now identified with 1 as well. An ant crawling to the right along the interval $[0, 1]$ will eventually come back to its starting point. This characterizes the topology (if not the geometry) of a circle. In formal language, $\mathbb{R}/\simeq$ is homeomorphic to a circle in the plane.

(iii) TYPO ALERT: The relation in (2a3) above is an equivalence relation, but I meant to define it as

$$(x, y) \sim (a, b) \text{ if and only if } (x, y) = (a, b) \text{ or } x = 0, a = 1, y + b = 1 \text{ or } x = 1, a = 0, y + b = 1.$$

As written in the original problem the set of equivalence classes is a folded rectangle. The relation I intended to write is described below.

We've taken the square and identified two opposite edges, albeit with a twist. Try taking a rectangular piece of paper and attaching one edge to the other. Without a twist you get a cylinder, but the relation $\sim$ has created something different. The set $[0, 1]^2/\sim$ is known as the Möbius strip.

(iv) The example is easier to see than the last one. We've identified all points on the boundary of the square together. This is essentially the same as the circle example above, but one higher dimension; the set $[0, 1]^2/\approx$ is a sphere. (Note: sphere refers to the surface; the interior of a sphere is called a ball).

(c) Consider the empty relation on a nonempty set S. For example, S could be the set of people on earth while the relation might be "has visited Jupiter with." This relation is symmetric and transitive because both properties state IF some objects are related, THEN something else happens. The empty relation satisfies those properties vacuously; however the relation is not reflexive.

The flaw in the proof is assuming that $x \sim y$ is true for some y; if x is related to nothing else, you cannot deduce reflexivity.

(d) Assume that $R \subseteq S \times S$ is both an equivalence relation and a function. Given any $x \in S$ we have $(x, x) \in R$ by reflexivity. Since R is a function, there can be no other ordered pairs within R—each $x \in S$ has already been 'sent' to a destination within S. Therefore $R = \{(x, x) : x \in S\}$. As a relation we know R as equality; as a function we call R the identity map $S \to S$.

(e) TYPO ALERT: This problem is nonsense as written. I'm really sorry about that. It was supposed to refer to the fact that a function's antiderivative has an arbitrary constant attached; that is, each equivalence class "looks like" a copy of $\mathbb{R}$. The collection of all classes however is quite large indeed—too large to be bijective with $\mathbb{R}$.

3 Constructing the Rational Numbers

(a) *Proof.* Let $(x, y) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Then $xy = xy$, whence $(x, y) \simeq (x, y)$.

Let $(x, y), (a, b) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$ satisfy $(x, y) \simeq (a, b)$. Then $ay = bx$, whence $bx = ay$ and $(a, b) \simeq (x, y)$.

Let $(x, y), (a, b), (w, z) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$ satisfy $(x, y) \simeq (a, b)$ and $(a, b) \simeq (w, z)$. Then both $ay = bx$ and $az = bw$, so we have $bxz = ayz = bwy$. This gives $b(xz - wy) = 0$; since $b \neq 0$ we conclude $xz = wy$ and $(x, y) \simeq (w, z)$. □

(b) *Proof.* Assume $(a, b) \simeq (c, d)$ and $(x, y) \simeq (z, w)$. That is, $ad = bc$ and $xw = yz$. Then we have

$$(ay + bx)dw = (ad)yw + bd(xw) = (bc)yw + bd(yz) = by(cw + dz),$$

so that $(ay + bx, by) = (cw + dz, dw)$, as desired. $\square$

(c) *Proof.* The subtle part of this is the first step. The function is not well-defined a priori; we would need to check that if $[a, b] = [c, d]$ then $f([a, b]) = f([c, d])$. This follows easily (fortunately) since $ad = bc$ implies

$$f([a, b]) = \frac{a}{b} = \frac{c}{d} = f([c, d]).$$

The function f is clearly surjective—given an arbitrary $a/b \in \mathbb{Q}$ simply note that $f([a, b]) = a/b$. Injectivity is slightly more involved. Assume that $f([a, b]) = f([c, d])$ for some $[a, b], [c, d] \in Q$. That is, $a/b = c/d$. This implies $ad = bc$, so that $[a, b] = [c, d]$. We conclude that f is a bijection.

Finally, take arbitrary $[a, b], [x, y] \in Q$. Note that

$$f([a, b] + [x, y]) = f([ay + bx, by]) = \frac{ay + bx}{by} = \frac{a}{b} + \frac{x}{y} = f([a, b]) + f([x, y]),$$

as desired. $\square$