1. Give an example of $A, B \subset \mathbb{R}$ with $A \neq B$ and $d(A, B) = 0$.
   Any sets $A \subset B$ where $B \setminus A$ has measure zero will suffice. For example, take $A = \mathbb{R} \setminus \mathbb{Q}$ and $B = \mathbb{R}$.

2. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous then $f$ is Lebesgue measurable.
   Let $a \in \mathbb{R}$. Then $\{x : f(x) > a\}$ is the inverse image under $f$ of the open set $(a, \infty)$; by continuity, $\{x : f(x) > a\}$ is open. All open sets are measurable, so $\{x : f(x) > a\}$ is measurable. This proves $f$ is a measurable function.

3. If $\{f_n\}$ is a sequence of measurable functions, prove that the set of $x$ for which $\{f_n(x)\}$ converges is measurable.
   The set whereupon $(f_n)$ converges is $\{x : \lim \inf f_n(x) = \lim \sup f_n(x)\}$.
   Define the measurable functions $g = \lim \inf f_n$ and $h = \lim \sup f_n$. Then $h - g$ is measurable, so 
   $\{x : \lim \inf f_n(x) = \lim \sup f_n(x)\} = \{x : 0 = h(x) - g(x)\} \setminus \{x : 0 < h(x) - g(x)\}$
   is measurable by theorem 11.15.
   Alternatively,
   $$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \left\{ x : |f_n(x) - f_m(x)| < \frac{1}{k} \right\}$$
   is exactly the set whereupon $(f_n)$ converges.

4. Is the Cantor set $C$ measurable?
   Let $C_0 = [0, 1]$ and $C_1$ be $C_0$ with its middle-third interval deleted—that is, $C_1 = C_0 \setminus (1/3, 2/3)$. For $n \geq 2$ define $C_n$ to be $C_{n-1}$ with the middle-third of each interval deleted. By definition, $C = \cap_n C_n$. Notice that each $C_n$ is an elementary set and
   $$d(C, C_n) = m(C \Delta C_n) \leq m(C_n) = \left(\frac{2}{3}\right)^n,$$
   where we use the common notation $A \Delta B$ to denote the symmetric difference of $A$ and $B$. Therefore $\lim_n d(C, C_n) = 0$, showing that $C$ is measurable.

5. What is $m^*(C)$?
   Since $C \subset C_n$ for each $n$, we have
   $$m^*(C) \leq m^*(C_n) = \left(\frac{2}{3}\right)^n,$$
   so taking $n \to \infty$ gives $m^*(C) = 0$.
   (Remark: All sets of outer measure zero are measurable; hence this problem answers the previous one)

6. Write a paragraph about Hausdorff measure and Hausdorff dimension.
   In the concrete approach to Lebesgue measure on $\mathbb{R}^n$ one covers a set $S$ with small sets and sums their volumes to approximate the ‘volume’ of $S$. Inspired by this, we note that the notion of diameter exists in all metric spaces, so perhaps a generalization of measure possible. In computing the volume of a ball in
In \( \mathbb{R}^n \), one raises the radius to the \( n \)-th power; how do we adapt this for a general space? We sidestep the problem; let \( S \) be a subset of a metric space \((X, d)\) and define for each \( r > 0 \)

\[
H^r(S) = \sup_{\delta > 0} \left\{ \sum_{i=1}^{\infty} \text{diam}(U_i)^r \right\},
\]

where the infimum is over all countable covers of \( S \) by set \( U_i \) with diameter less than \( \delta \) (recall that \( \text{diam}(U) = \sup\{d(x, y) : x, y \in U\} \)). It can be shown that for each \( r \) this is an outer measure; better yet, in \( \mathbb{R}^n \) Lebesgue measure agrees with \( H^n \) up to a constant.

In \( \mathbb{R}^2 \) the measure of a line is zero. In some sense this is because the line has ‘dimension’ smaller than that of \( \mathbb{R}^2 \), so the measure associated with \( \mathbb{R}^2 \) returns zero. The same phenomenon occurs for our general measure: given a set \( S \) in a metric space, taking \( r \) too large yields \( H^r(S) = 0 \). This suggests a definition of dimension far more general than that of linear algebra. Define the dimension of \( S \) to be

\[
\dim(S) = \inf\{r \geq 0 : H^r(S) = 0\}.
\]

Amazingly, this definition matches our expectations for simple sets. For instance, \( \dim(\mathbb{R}^n) = n \) and \( \dim(S^1) = 1 \). Stranger still, some set have dimension which is not a whole number! The Cantor set has dimension \( \ln 2 / \ln 3 \), while Brownian motion traces a path with dimension between 1 and 2.