# Math 8 Homework 5 Solutions

## 1 Mathematical Induction and the Well Ordering Principle

(a) *Proof.* When $n = 1$ we have

$$1 + 3 + 5 + \cdots + (2n-1) = 1 = n^2.$$

Now assume the claim holds for some positive integer $n$. Then we have that

$$1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = n^2 + (2n+1) = (n+1)^2$$

so the result holds by induction. □

(b) *Proof.* When $n = 1$ we have

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = 1 = 2 - 1 = (n+1)! - 1.$$

Now assume the result for some positive integer $n$. Then it follows that

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n+1) \cdot (n+1)! &= (n+1)! - 1 + (n+1) \cdot (n+1)! \\ &= (n+2) \cdot (n+1)! - 1 \\ &= (n+2)! - 1, \end{aligned}$$

so the result holds for all $n$ by induction. □

(c) *Proof.* When $n = 1$ we have $4^n - 1 = 3$, a multiple of 3. Now assume that 3 divides $4^n - 1$ for some positive integer $n$. Find an integer $k$ so that $4^n - 1 = 3k$ and note that

$$4^{n+1} - 1 = 4 \cdot 4^n - 1 = 3 \cdot (4^n + k).$$

We see that 3 divides $4^{n+1} - 1$. By induction the result holds for all integers $n$. □

(d) *Proof.* When $n = 1$ we have that

$$\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots = e - 2 < 1 = \frac{1}{n!}.$$

Now assume the result holds for some positive integer $n$. Then it follows that

$$\begin{aligned} \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots &= \frac{1}{(n+2)(n+1)!} + \frac{1}{(n+3)(n+2)!} + \frac{1}{(n+4)(n+3)!} + \cdots \\ &< \frac{1}{(n+1)(n+1)!} + \frac{1}{(n+1)(n+2)!} + \frac{1}{(n+1)(n+3)!} + \cdots \\ &= \frac{1}{n+1}\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots\right) \\ &< \frac{1}{(n+1)n!} \\ &= \frac{1}{(n+1)!}, \end{aligned}$$

so the result holds for all $n$ by induction. □

(e) *Proof.* We proceed by induction on $n$. When $n = 1$ we have $x + 1/x \in \mathbb{Q}$ by assumption. We also need to handle $n = 2$ separately; in this case notice that

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 \in \mathbb{Q}.$$

Now assume that $n \geq 2$ is an integer so that the result holds for both $n$ and $n-1$ (that is, we're assuming the result for the previous two values of $n$—this is why we had two base cases). We can deduce the result for $n+1$ by noting that

$$x^{n+1} + \frac{1}{x^{n+1}} = \left(x^n + \frac{1}{x^n}\right)\left(x + \frac{1}{x}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right) \in \mathbb{Q},$$

completeing the induction. □

(f) *Proof.* This is tricky, but it's a well–known and important problem. First notice that $f(0) = f(0 + 0) = f(0) + f(0)$, forcing $f(0) = 0$. From this we deduce that for any $x \in \mathbb{Q}$ it follows that

$$0 = f(0) = f(x - x) = f(x) + f(-x),$$

so $f(-x) = -f(x)$. In other words, once we prove $f(x) = cx$ for $x > 0$ it follows for $x < 0$; from now on we'll only consider $x > 0$.

Let $f(1) = c$. We'll prove $f(n) = cn$ for all $n \in \mathbb{N}$ by a brief induction. The base case is immediate, and if we assume $f(n) = cn$ for some $n \in \mathbb{N}$ then we have $f(n + 1) = f(n) + f(1) = cn + c = c(n + 1)$, proving the first claim.

Next we'll prove $f(x) = cx$ for any $x$ of the form $1/n$, where $n$ is a positive integer. Given any such $n$ we have

$$c = f(1) = f\Big(\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{n}\Big) = nf\left(\frac{1}{n}\right),$$

so that $f(1/n) = c \cdot (1/n)$, as claimed.

Finally we show $f(x) = cx$ for general rational numbers $x$. Let $m, n > 0$ be integers. Then we have

$$f\left(\frac{m}{n}\right) = f\Big(\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{m}\Big) = mf\left(\frac{1}{n}\right) = c \cdot \frac{m}{n},$$

as desired. □

(g) *Proof.* The image of $f$ is a set of positive integers. By the well–ordering principle there is a smallest element. Let $n \in \mathbb{Z}$ be such that $f(n)$ is minimal. If $f(n + 1) \neq f(n)$ then $f(n + 1) > f(n)$ since $f(n)$ is the smallest possible value of $f$. But then

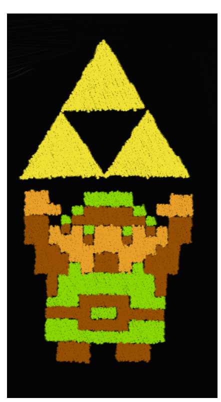
$$f(n - 1) = 2f(n) - f(n + 1) < f(n),$$

contradicting the fact that $f(n)$ is minimal. Thus $f(n + 1) = f(n)$ and similarly $f(n - 1) = f(n)$. An inductive argument (whose details we need not provide) shows that $f(n + k) = f(n - k) = f(n)$ for any $k$ and hence $f$ is constant. □

## 2 The Pigeonhole Principle

(a) *Proof.* There are 100 possibilities for how many hands a person can shake. If no two people shook the same number of hands, then each person shook a different number of hands. However, this means someone shook 99 hands (everyone else's hand) and someone shook no hands (not even the 99 guy), which is a contradiction. So some two people did shake the same number of hands. □

(b) *Proof.* Subdivide the triangle into 4 equilateral triangles as shown:



If there are 5 points chosen in this figure, some two are in the same smaller triangle. These two points are at most $1/2$ apart. □

(c) *Proof.* Rewrite the 7 numbers as $\tan\theta_i$, where $\theta_1, \dots, \theta_7 \in (-\pi/2, \pi/2)$. This interval has length $\pi$, so among 7 numbers there must be some two at most $\pi/6$ apart. That is, we can find distinct $i, j \in \{1, \dots, 7\}$ so that

$$0 \leq \theta_i - \theta_j \leq \frac{\pi}{6}.$$

The tangent function is increasing (that is, it preserves inequalities). Since $\tan 0 = 0$ and $\tan \pi/6 = 1/\sqrt{3}$, we have

$$0 \leq \tan(\theta_i - \theta_j) \leq \frac{1}{\sqrt{3}}.$$

Using the identity for tangent of a difference of angles gives

$$\tan(\theta_i - \theta_j) = \frac{\tan\theta_i - \tan\theta_j}{1 + \tan\theta_i \tan\theta_j} = \frac{x_i - x_j}{1 + x_i x_j},$$

which gives the result. □

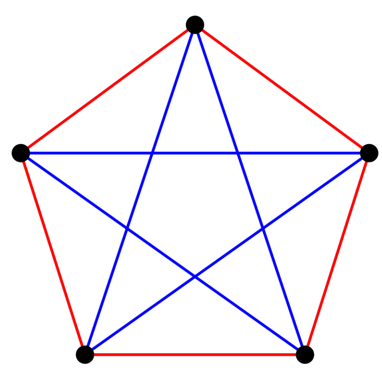
(d) *Proof.* Ramsey sees 5 other people in the room, so some three of them must be of the same relationship to him. Without loss of generality, say Ramsey has at least 3 friends in the room. If any two of those friends are friends with each other, together with Ramsey they are a trio of friends. Otherwise, every pair of Ramsey's friends are strangers to each other, giving us a trio of strangers. □

(e) *Proof.* Ramsey sees 16 other people in the room, so there must be 6 of the same relationship to him. Without loss of generality, say Ramsey has at least 6 archenemies in the room. If any pair of the 6 people are archenemies with each other, then together with Ramsey we have a trio of enemies. Otherwise we have 6 people, any two of which are friends or strangers. By the previous problem we know there must be a trio of friends or a trio of strangers in any such group, so we are done. □

Remark: This 'reduce to a known problem and stop' type of argument is a well–known quirk of mathematical reasoning. Too many silly jokes told by physicists are based on this.

(f) *Proof.* We've already shown that among 6 people some 3 must be mutual friends or strangers. That is, $R_2 \leq 6$. We're not done because we haven't checked whether or not 5 people would suffice to guarantee the same result. In other words, we need only provide an example wherein no trios of mutual friends or strangers exist among 5 people.

Consider the following picture:



If each vertex represents a person, each red edge represents friendship, and each blue edge represents strangeness, then a trio of mutual friends or strangers would be a monochromatic triangle. None exists in this picture, so we've shown that 5 people is not enough to *guarantee* a trio of mutual friends or strangers. That is, $R_2 > 5$ and we are done. □

(g) *Proof.* We'll use an argument which should be becoming familiar by now. For simplicity we'll refer to a trio of people with the same relationship type simply as a trio. Suppose $(n+1)(R_n - 1) + 2$ people are in a room together, any two of which has a relationship from one of $n+1$ types. If one of them is named Ramsey, then Ramsey sees $(n+1)(R_n - 1) + 1$ other people. We claim that at least $R_n$ people of one type of relationship to Ramsey are in the room with him. This follows from the pigeonhole principle, but for clarity we'll give details. If there weren't $R_n$ people of the same relationship to Ramsey, then there could be at most

$$\underbrace{(R_n - 1) + (R_n - 1) + \cdots + (R_n - 1)}_{n+1} = (n+1)(R_n - 1)$$

other people in the room, which is a contradiction. This proves the claim.

Without loss of generality, say those $R_n$ people are friends to Ramsey. If any two of them are friends with each other, together with Ramsey we have a trio. Otherwise, we have $R_n$ people, any two of which have one of $n$ relationships to each other (since we've ruled out 1 type of relationship). By definition of $R_n$ there must be a trio in this group of people. Therefore $(n+1)(R_n - 1) + 2$ people is enough to guarantee a trio when there are $n + 1$ types of relationships. This proves the result. □

(h) Before we begin, we need a lemma.

**Lemma.** *For any integer $n$,*

$$\lfloor n!e \rfloor = n! \left( \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!} \right).$$

*Proof.* First notice that

$$n! \left( \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!} \right)$$

is an integer since each term $n!/k! = n(n-1)\cdots(k+1)$ is an integer. Secondly,

$$n! \left( \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \right) < 1$$

by one of the problems at the beginning of the homework. Since

$$e = \frac{1}{0!} + \frac{1}{1!} + \cdots,$$

the result follows. □

*Proof.* We'll proceed by induction on $n$. When $n = 2$ we have

$$R_2 = 6 \leq \lfloor 2e \rfloor + 1.$$

Now assume the result holds for some integer $n \geq 2$. Then we have that

$$R_{n+1} \leq (n+1)(R_n - 1) + 2 \leq (n+1)\lfloor n!e \rfloor + 2.$$

Using the lemma we can write

$$\begin{aligned} R_{n+1} &\leq (n+1) \cdot n! \left( \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!} \right) + 2 \\ &= (n+1)! \left( \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!} \right) + \frac{(n+1)!}{(n+1)!} + 1 \\ &= \lfloor (n+1)!e \rfloor + 1, \end{aligned}$$

so the result holds for all $n$ by induction. □