Math 8 Homework 6 Solutions

1 Clever Counting

(a) Each element of $\{1, 2, \ldots, n\}$ must be placed in at least one of the three sets, although not all 3 at once. Hence each element can be placed in the sets A, B, C in 6 ways: A and B (not C), B and C (not A), C and A (not B), A only, B only, or C only. There are 6 choices made independently for n objects, so the answer is 6^n.

(b) *Proof.* We have n^2 people in our marching band. We wish to arrange them into an $n \times n$ square marching formation, but we only care which row a person is in (not which column). There are $(n^2)!$ ways to arrange the people, but we divide by $n!$ for each row (since we consider all its permutations the same). There are n rows, so the number of formations we can make is $(n^2)!/(n!)^n$. In particular, $(n^2)!/(n!)^n$ is an integer. □

(c) (i) We could proceed symbolically:

$$\sum_{A \subseteq S} |A| = \sum_{A \subseteq S} \sum_{x \in A} 1 = \sum_{x \in S} \sum_{\substack{A \subseteq S \\ A \ni x}} 1 = \sum_{x \in S} 2^{n-1} = n2^{n-1}.$$

Or rather, we could reason cominatorially. This sum counts elements in subsets of S. Each element is counted in this sum once for every subset it appears in. There are 2^{n-1} such subsets, so each element is counted that many times. There are n elements, so the total is $n2^{n-1}$.

(ii) We'll only give the combinatorial argument. This sum counts the elements in subsets of S. Each element appears 4^{n-1} times (2^{n-1} choices for A and B, independently made), so the total is $n4^{n-1}$.

(d) Given positive integers n, k with $k \leq n$, the Stirling number of the second kind $S(n, k)$ is defined to be the number of ways to place n balls into k identical boxes, leaving no box empty.

(i) To place n things into 1 box, there is only one way (just do it). Hence $S(n, 1) = 1$.

To place n things into 2 boxes, there are 2 choices for each object, giving 2^n arrangements. However, 2 possibilities leave one box empty (all objects in a single box), so we have only $2^n - 2$ arrangements. Finally we consider the boxes indistinguishable, so division by 2 gives $S(n, 2) = 2^{n-1} - 1$.

To place n objects into n boxes, there is only one way (as boxes are indistinguishale, there is no choice to make). Hence $S(n, n) = 1$.

To place n objects into $n - 1$ boxes, there is only the choice of which 2 objects are placed together. Hence $S(n, n-1) = \binom{n}{2}$.

(ii) *Proof.* Consider placing n objects into k indistinguishable boxes, leaving none empty. There are two distinct cases: whether or not object n is placed in its own box. First we could place $n - 1$ objects into $k - 1$ boxes, leaving the final object 1 choice for its box. This is done in $S(n-1, k-1)$ ways. Second we could place $n - 1$ things into k boxes in $S(n - 1, k)$ ways before choosing a destination for the final object in k ways. The total number of arrangements in this situation is $kS(n - 1, k)$. Finally $S(n, k)$ is the sum of these two counts, giving the result. □

(iii) *Proof.* When $n = 3$ we have

$$S(n, n-2) = S(3, 1) = 3 = \binom{3}{0} + 3\binom{3}{4} = \binom{n}{3} + 3\binom{n}{4}.$$

Now assume the result for some integer $n \geq 3$. Then we have

$$\begin{aligned} S(n+1, n-1) &= S(n, n-2) + (n-1)S(n, n-1) \\ &= \binom{n}{3} + 3\binom{n}{4} + (n-1)\binom{n}{2} \\ &= \binom{n+1}{3} + 3\binom{n+1}{4}, \end{aligned}$$

so the result follows from induction on n. □

(e) *Proof.* Let the answer be denoted α. We proceed symbolically:

$$\begin{aligned}\alpha &= (2^n-1)^{-1} \sum_{\substack{A\subseteq S\\ A\neq\varnothing}} \sum_{x\in A} \frac{x}{|A|} \\ &= (2^n-1)^{-1} \sum_{x\in S} \sum_{A\ni x} \frac{x}{|A|} \\ &= (2^n-1)^{-1} \sum_{x\in S} x \sum_{A\ni x} \frac{1}{|A|}. \qquad (\star)\end{aligned}$$

To evaluate this inner sum we ask the question: How many k–element subsets of S contain the element x? Such a subset contains x and $k-1$ other elements chosen from $n-1$ numbers, so $\binom{n-1}{k-1}$ is the answer to this query. We decompose the inner sum of equation $(\star)$ into a sum over sets of different sizes:

$$\sum_{A\ni x} \frac{1}{|A|} = \sum_{k=1}^{n} \sum_{\substack{A\ni x\\ |A|=k}} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k}\binom{n-1}{k-1} = \sum_{k=1}^{n} \frac{1}{n}\binom{n}{k} = \frac{2^n-1}{n}.$$

The -1 is in the numerator since $k=0$ was omitted from the well-known sum $\binom{n}{0}+\cdots+\binom{n}{n}=2^n$. Inserting this into equation $(\star)$ gives

$$\alpha = (2^n-1)^{-1} \sum_{x\in S} x \left(\frac{2^n-1}{n}\right) = \frac{1}{n}\sum_{x\in S} x,$$

which is simply the arithmetic mean of the elements of S. □

(f) *Proof.* For each $m \in S$ we count how many k–element subsets contain m as the smallest element. Any such subset must contain m and $k-1$ other elements chosen from $\{m+1, m+2, \ldots, n\}$. This set contains $n-m$ elements, so there are $\binom{n-m}{k-1}$ subsets with m as the smallest element. Note that only $m = 1, 2, \ldots, n-k+1$ need to be considered. Let α be the desired average and $\mathrm{sm}(A)$ denote the smallest element of A; then we have

$$\alpha = \binom{n}{k}^{-1} \sum_{\substack{A\subseteq S\\ |A|=k}} \mathrm{sm}(A) = \binom{n}{k}^{-1} \sum_{m=1}^{n-k+1} m\binom{n-m}{k-1}$$

Evaluating this last sum can be tricky. We'll use a well–known result called the Hockey Stick Identity:

Lemma. *Let m, n be positive integers. Then*

$$\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+m}{n} = \binom{n+m+1}{n+1}.$$

Proof of lemma. There are many proofs of this identity, but since we're assuming no prior knowledge of combinatorics we'll use induction on m. When $m = 0$ the identity is obviously true. Now suppose it holds for some positive integer m. Then we have

$$\binom{n}{n} + \binom{n+1}{n} + \cdots + \binom{n+m}{n} + \binom{n+m+1}{n} = \binom{n+m+1}{n+1} + \binom{n+m+1}{n} = \binom{n+m+2}{n+1},$$

where the last equality follows from Pascal's identity (problem 2b on this homework). By induction the result holds for all m, proving the lemma. □

Now we return to the original problem. Rewriting the sum in a clever way and using the lemma twice yields

$$\begin{aligned}
\alpha &= \binom{n}{k}^{-1} \sum_{m=1}^{n-k+1} \sum_{j=1}^{m} \binom{n-m}{k-1} \\
&= \binom{n}{k}^{-1} \sum_{j=1}^{n-k+1} \sum_{m=j}^{n-k+1} \binom{n-m}{k-1} \\
&= \binom{n}{k}^{-1} \sum_{j=1}^{n-k+1} \left[\binom{k-1}{k-1} + \binom{k}{k-1} + \cdots + \binom{n-j}{k-1}\right] \\
&= \binom{n}{k}^{-1} \sum_{j=1}^{n-k+1} \binom{n-j+1}{k} \\
&= \binom{n}{k}^{-1} \left[\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k}\right] \\
&= \binom{n}{k}^{-1} \binom{n+1}{k+1} \\
&= \frac{n+1}{k+1},
\end{aligned}$$

as desired. □

2 Binomial Identities

(a) *Proof.* Billy has n friends and wants to k of them to the amusement park. Choosing which k to bring (in $\binom{n}{k}$ ways) is the same as choosing which $n-k$ to leave behind (in $\binom{n}{n-k}$ ways). □

(b) *Proof.* Billy wants to take $k+1$ of his $n+1$ friends to the amusement park. He could make his decision in $\binom{n+1}{k+1}$ ways. Alternatively since one of Billy's friends, Alice, always wear a goofy red hat, Billy could decide first whether to take Alice before considering the rest of his friends. If he takes Alice, he chooses k more people from the remaining n in $\binom{n}{k}$ ways. If he doesn't bring Alice, he needs to choose all $k+1$ people from the remaining n in $\binom{n}{k+1}$ ways. Altogether the number of choices is $\binom{n}{k} + \binom{n}{k+1}$. □

(c) *Proof.* Billy want to take k of his n friends to the park for his birthday. Furthermore, he wants m of the partygoers to wear birthday hats. He could choose the attendees in $\binom{n}{k}$ ways before choosing the hat–wearers in $\binom{k}{m}$ ways, for a total count of $\binom{n}{k}\binom{k}{m}$. Another way Billy could invite people is by first choosing which m of his friends he wants to see wear hats. He has $\binom{n}{m}$ choices for this. These m people go to the park, but Billy needs $k-m$ more people chosen from the remaining $n-m$ to attend as well. He can make this choice in $\binom{n-m}{k-m}$ ways, for an altogether count of $\binom{n}{m}\binom{n-m}{k-m}$. □

(d) *Proof.* Billy wants to take some number of his n friends to the park. He could simply go to each friend separately and decide whether to take her (2 choices, repeated n times) giving 2^n total choices for groups to invite. Alternatively Billy could arbitrarily decide to invite k people (for some $k = 0, 1, \ldots, n$) and pick a group of size k in $\binom{n}{k}$ ways. Summing this over all k gives the number of choices Billy has in inviting friends. □

(e) *Proof.* Billy is yet again going to the park and taking some number of his n friends. This time however he wants one attendee to wear a big goofy hat. He could choose the hat wearer first in n ways before picking the rest of the group in 2^{n-1} ways. In total there are $n2^{n-1}$ choices for Billy. Alternatively, Billy could decide to take k friends for some $k = 1, \ldots, n$ in $\binom{n}{k}$ ways before choosing one of those k people to wear the hat. For each k Billy has $k\binom{n}{k}$ choices; summing over k gives the total number of ways to invite friends, one of whom wears a hat. □

(f) *Proof.* Billy has reached an awkward time in his life and keeps two separate groups of friends, one with n people and one with m. He's decided to mix company and take r people to the park for his birthday. He could simply lump the friends together and make one of $\binom{n+m}{r}$ choices to bring people to the park. Alternatively, Billy could arbitrarily decide that k people should attend from his first group of friends, while $r-k$ are chosen from the second group. Given a value of k, Billy has $\binom{n}{k}\binom{m}{r-k}$ choices; summing over k gives all possibilities for inviting r people. $\square$

(g) *Proof.* Billy has n guy friends and n girl friends. He wants to take n of them to the park, but insists that one of the guys wear a big goofy hat. He could choose the hat wearer in n ways before choosing the rest of the attendees in $\binom{2n-1}{n-1}$ ways, giving a total count of $n\binom{2n-1}{n-1}$. Alternatively Billy could decide that k guys should attend for some $k = 1, \ldots, n$. After choosing the k guys in $\binom{n}{k}$ ways, he chooses a guy to wear the hat in k ways. To round out his entourage Billy picks k girls to go home (in $\binom{n}{k}$ ways), taking the rest with him to the park. Given a value of k, Billy has $k\binom{n}{k}^2$ choices; summing over k gives all ways of building the necessary group of friends. $\square$