Math 118C Homework 7 Solutions
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1. If $f \in L^1(\mu)$ and $g$ is bounded and measurable on $E$, prove that $fg \in L^1(\mu)$.
   
   Let $M > 0$ be such that $|g| \leq M$. Then
   \[ \int_X |fg| \, d\mu \leq \int_X M|f| \, d\mu < \infty. \]

2. Let $g(x) = \chi_{(1/2,1]}$. Define $f_{2k}(x) = g(x)$ and $f_{2k+1}(x) = g(1-x)$ on $[0,1]$. Show that $\lim \inf f_n = 0$ but $\int f_n = 1/2$ for each $n$.

   For $x \in [0,1/2]$, we find that $f_n(x) = (1+(-1)^{n+1})/2$. For $x \in (1/2,1]$, we find that $f_n(x) = (1+(-1)^n)/2$. Either way, $\lim \inf f_n(x) = 0$. That $\int f_n = 1/2$ is clear.

3. Let $f_n = (1/n)\chi_{[-n,n]}$. Then $f_n \to 0$ uniformly on $\mathbb{R}$ but $\int f_n = 2$ for each $n$. On sets of finite measure uniform convergence of bounded functions do allow interchange of limit and integral.

   Let $\epsilon > 0$ and find $N > 0$ so that $1/N < \epsilon$. Then for $n \geq N$ and $x \in \mathbb{R}$,
   \[ |f_n(x)| \leq 1/n < 1/N < \epsilon. \]

   Thus $f_n$ converges uniformly to zero. That $\int f_n = 2$ for each $n$ is obvious.

   Suppose that $X \subset \mathbb{R}$ has finite measure and $g_n \to g$ uniformly on $X$, with the functions $g_n$ bounded.
   \[ \left| \int_X (g_n - g) \, d\mu \right| \leq \sup_X |g_n - g| \int_X \, d\mu = \sup_X |g_n - g| \mu(X). \]

   If $\mu(X) = 0$ then it follows that $\int g_n \to \int g$: assume that $\mu(X) > 0$. Let $\epsilon > 0$ and find $N > 0$ so that $\sup_X |g_n - g| < \epsilon/\mu(X)$ for all $n \geq N$. Then for all such $n$
   \[ \left| \int_X (g_n - g) \, d\mu \right| \leq \sup_X |g_n - g| \mu(X) < \epsilon. \]

   Thus $\int g_n \to \int g$.

4. If $\mu(X) < \infty$ and $f \in L^2(\mu)$, then $f \in L^1(\mu)$.

   By the Cauchy-Schwarz inequality (theorem 11.35),
   \[ \int_X |f| \, d\mu = \int_X |f| \cdot 1 \, d\mu \leq \left( \int_X |f|^2 \right)^{1/2} \left( \int_X 1^2 \right)^{1/2} = \|f\| \sqrt{\mu(X)} < \infty. \]

5. If $f, g \in L^1(\mu)$ define the distance function to be
   \[ d(f, g) = \int_X |f - g| \, d\mu. \]

   Prove that $L^1(\mu)$ is a complete metric space (identifying functions that agree almost everywhere).

   Clearly $L^1$ is a vector space over $\mathbb{C}$. The metric $d$ is certainly nonnegative and symmetric. Suppose that $d(f, g) = 0$; from problem 11.1 this implies $f = g$ almost everywhere. Under our equivalence relation, we regard $f$ and $g$ as equal, so $d(f, g) = 0$ implies $f = g$ in $L^1$. The triangle inequality follows from that of the absolute value and the order preservation of the integral.
Let \((f_n)\) be a Cauchy sequence with respect to the metric \(d\). To show that \((f_n)\) converges, it suffices to show that a subsequence converges; proof of this fact is a standard exercise. Passing to a subsequence as needed, assume that
\[
d(f_n, f_{n+1}) \leq 2^{-n}
\]
for each \(n\). From theorem 11.30 we can write
\[
\int_X \left| f_n - f_{n+1} \right| \, d\mu = \sum_{n=1}^{\infty} \int_X |f_n - f_{n+1}| \, d\mu < \sum_{n=1}^{\infty} 2^{-n} < \infty.
\]
It follows that the sum \(\sum_n |f_n - f_{n+1}|\) must converge almost everywhere on \(X\). This implies convergence of \(f_N(x) = f_1(x) + \sum_{n=1}^{N} (f_{n+1}(x) - f_n(x))\)
for almost all \(x \in X\). Set \(f(x) = \lim f_n(x)\), defining \(f\) arbitrarily on the remaining set of measure zero. Then \(f_n \to f\) in \(L^1\), implying that \(L^1\) is complete.

6. For \(n \in \mathbb{N}\) and \(x \in [-\pi, \pi]\) define \(f_n(x) = \sin nx\). Show that the set \(\{f_n\}\) is closed and bounded in \(L^2\) but not compact.

For any \(n \in \mathbb{N}\) we have
\[
\|\sin nx\|^2 = \int_{-\pi}^{\pi} |\sin nx|^2 \, dx = \frac{1}{n} \int_{-\pi}^{\pi} 1 - \cos 2mu \, du = \pi.
\]
Thus the set \(\{f_n\}\) is bounded in \(L^2\) norm. If \(m \neq n\) then
\[
\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \int_{-\pi}^{\pi} \frac{\cos(m-n)x - \cos(m+n)x}{2} \, dx = 0.
\]
Thus the set \(\{f_n/\sqrt{\pi}\}\) is orthonormal. Also, if \(m \neq n\) another computation gives
\[
\|f_n - f_m\|^2 = \int_{-\pi}^{\pi} |\sin nx - \sin mx|^2 \, dx = \int_{-\pi}^{\pi} (|\sin nx|^2 - 2 \sin nx \sin mx + |\sin mx|^2) \, dx = 2\pi.
\]
Suppose that \(f_{n_k}\) is convergent in \(L^2\) norm; then \((f_{n_k})\) is a Cauchy sequence. From the previous computation, we see that \((f_{n_k})\) must be eventually constant, hence convergent to an element of the set \(\{f_n\}\). This shows that \(\{f_n\}\) is a closed set.

From Bessel’s inequality we have for any \(\phi \in L^2[-\pi, \pi]\),
\[
\sum_{n=1}^{\infty} \left| \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin(nx) \overline{\phi(x)} \right| \leq \int_{-\pi}^{\pi} |\phi|^2,
\]
and in particular,
\[
\left| \int_{-\pi}^{\pi} f_n(x) \overline{\phi(x)} \, dx \right| \to 0
\]
as \(n \to \infty\).

Assume that the set \(\{f_n\}\) is compact. Then the sequence \((f_n)\) must have a convergent subsequence \((f_{n_k})\); denote \(g = \lim_k f_{n_k}\). Notice that for any \(\phi \in L^2[-\pi, \pi]\),
\[
\left| \int_{-\pi}^{\pi} (f_{n_k}(x) - g(x)) \overline{\phi(x)} \, dx \right| \leq \|f_{n_k} - g\| \|\phi\| \to 0
\]
as \(k \to \infty\). We have the relation
\[
\|g\|^2 = \int_{-\pi}^{\pi} g(x) \overline{g(x)} \, dx = \int_{-\pi}^{\pi} f_{n_k}(x) \overline{g(x)} \, dx - \int_{-\pi}^{\pi} (f_{n_k}(x) - g(x)) \overline{g(x)} \, dx,
\]
so taking \(k \to \infty\) gives \(g = 0\) in \(L^2\). We conclude that
\[
\pi = \|f_{n_k}\|^2 = \|f_{n_k} - g\|^2 \to 0
\]
as \(k \to \infty\). This is a contradiction, so \(\{f_n\}\) is not a compact set.