Math 8 Homework 7 Solutions

1 Cardinality and Countability

(a) (i) *Proof.* Given an even positive integer n define $f(n) = n - 1$. Clearly this function is defined for all even integers and is injective. Surjectivity follows since every odd positive integer is 1 less than an even positive integer. □

(ii) *Proof.* Let $f : \mathbb{R} \to (0,1)$ be given by $f(x) = 1/2 + (4/\pi)\arctan x$. We know that f is injective since f' is never 0. Since $\arctan x$ takes all values from $-\pi/2$ to $\pi/2$, our function f is surjective. Thus f is the required bijection. □

(iii) *Proof.* Define $f : (0,1) \to (0,1]$ as follows: If $n \geq 2$ is an integer, then let $f(1/n) = 1/(n-1)$. For all $x \in (0,1)$ which are not reciprocals of integers, let $f(x) = x$. We've defined f on the entire domain, and it takes values in $(0,1]$. The function is surjective because any x which is not a reciprocal of an integer is obtained from $f(x) = x$; meanwhile a number of the form $1/n$ for $n \in \mathbb{N}$ is obtained via $f(1/(n+1)) = 1/n$. Injectivity follows similarly. □

(b) *Proof.* This is a corollary of Cantor's theorem; given any set S the power set $\mathcal{P}(S)$ is strictly larger. □

(c) Which of the following sets are countable?

(i) $\mathbb{Q} \cap \{x \in \mathbb{R} : \sin x > 1/2\}$ is a subset of $\mathbb{Q}$, hence countable.

(ii) $\mathbb{R} - \mathbb{Q}$ is uncountable; if it weren't we'd have $\mathbb{R}$ is countable using $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$.

(iii) The set of atoms in the observable universe is finite, hence countable. Huge, but finite.

(iv) A circle is uncountable since it's just $(0,1]$, all rolled up. Yep, circles are bigger than the universe.

(v) There is a bijection from the set of all binary strings to $\mathcal{P}(\mathbb{N})$. Given a such a string define a set $S \in \mathcal{P}(\mathbb{N})$ as

$$S = \{n \in \mathbb{N} : \text{the } n\text{–th digit in the string is a } 1\}$$

An inverse function exists; given $S \in \mathcal{P}(\mathbb{N})$ make a string $x_1, x_2, x_3, \ldots$ via the rule

$$x_n = \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \notin S \end{cases}$$

Hence the aforementioned map is a bijection; since $\mathcal{P}(\mathbb{N})$ is uncountable, so is the set of binary strings.

(vi) There are 2^k binary strings of length k. If A_k denotes the set of all such strings, the set of finite binary is the (countable) union of the sets A_k. The set of finite binary strings is hence countable.

(d) *Proof.* For the base case $k = 1$ we already know that $\mathbb{Z}$ is countable. Now assume that $\mathbb{Z}^k$ is countable for some k. Note that $|\mathbb{Z}^{k+1}| = |\mathbb{Z}^k \times \mathbb{Z}|$ since there exists an obviously bijective map:

$$(x_1, x_2, \ldots, x_{n+1}) \mapsto ((x_1, x_2, \ldots, x_n), x_{n+1}).$$

We know $\mathbb{Z}^k \times \mathbb{Z}$ is countable, so $\mathbb{Z}^{k+1}$ is as well. Hence $\mathbb{Z}^k$ is countable for all positive integers k by induction. □

2 An Introduction to Measure

The length of an interval (a,b) is clear—we define $\ell(a,b) = b - a$ as expected—but we'd like to generalize length to more sets. Doing this in general is a bit complicated, so we'll restrict ourselves to studying zero length. Given a set $S \subseteq \mathbb{R}$ we say S has *measure zero* if for every $\epsilon > 0$ there is a countable collection of intervals $I_1, I_2, \ldots$ so that both $S \subseteq I_1 \cup I_2 \cup \cdots$ and

$$\ell(I_1) + \ell(I_2) + \cdots < \epsilon.$$

(a) *Proof.* Let $x \in \mathbb{R}$ and $\epsilon > 0$ be given. Then $(x - \epsilon/2, x + \epsilon/2)$ is an interval of length ϵ that covers $\{x\}$. □

(b) *Proof.* Let $\epsilon > 0$. Since B has measure zero, we can find a countable collection of intervals $I_1, I_2, \ldots$ which cover B and satisfy $\sum \ell(I_n) < \epsilon$. Since $A \subseteq B$, these intervals cover A and satisfy $\sum \ell(I_n) < \epsilon$, so A has measure zero. □

(c) *Proof.* Let $\epsilon > 0$ and find a countable collection of intervals $I_1, I_2, \ldots$ so that $A \subseteq \cup_n I_n$ and $\sum \ell(I_n) < \epsilon/2$. Similarly find a countable collection of intervals $J_1, J_2, \ldots$ so that $B \subseteq \cup_n J_n$ and $\sum \ell(J_n) < \epsilon/2$.

Together, these two collections of intervals form a single countable collection of intervals with both $A \cup B \subseteq (\cup_n I_n) \cup (\cup_n J_n)$ and $\sum \ell(I_n) + \sum \ell(J_n) < \epsilon$.

(To be explicit, we use the fact that a union of 2 countable collections is a countable collection) □

(d) *Proof.* The same argument from before works here, with a twist. Let $\epsilon > 0$. For each A_k find a countable collection of intervals $I_1^k, I_2^k, \ldots$ that cover A_k and satisfy $\sum_n \ell(I_n^k) < \epsilon/2^k$. (Note that here superscripts are being used as secondary indices, not exponents). We've approximated each set A_k with smaller and smaller error as k increases.

Since the union of a countable collection of countable collections is countable (what a mouthful), all together all the intervals (I_n^k) form a single countable collection of intervals which cover $\cup_n A_n$. Furthermore, the lengths satify

$$\sum_{n,k} \ell(I_n^k) = \sum_k \sum_n \ell(I_n^k) < \sum_k \frac{\epsilon}{2^k} = \epsilon.$$

Hence $\cup_n A_n$ has measure zero. □

(e) *Proof.* This is a corollary of the previous problem. Since singletons have measure zero and $\mathbb{Q}$ is a countable union of singletons, $\mathbb{Q}$ has measure zero. □