Math 8 Homework 8 Solutions

1. The answer is $\dfrac{6}{n(n+1)(2n+1)}$.

   Proof. By the Cauchy–Schwarz inequality,

   $$\begin{aligned} 1 &= (x_1 + 2x_2 + 3x_3 + \cdots + nx_n)^2 \\ &\le (1^2 + 2^2 + \cdots + n^2)(x_1^2 + x_2^2 + \cdots + x_n^2) \\ &= \left(\frac{n(n+1)(2n+1)}{6}\right)(x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2), \end{aligned}$$

   If $(x_1, x_2, \ldots, x_n)$ is parallel to $(1, 2, \ldots, n)$ then equality is achieved. The answer follows. □

2. *Proof.* Since $x, y, z > 0$ the roots $\sqrt{x}, \sqrt{y}, \sqrt{z}$ are real numbers. By Cauchy–Schwarz we have

   $$\begin{aligned} \frac{1}{x} + \frac{4}{y} + \frac{9}{z} &= \left(\sqrt{x}^2 + \sqrt{y}^2 + \sqrt{z}^2\right)\left(\left(\frac{1}{\sqrt{x}}\right)^2 + \left(\frac{2}{\sqrt{x}}\right)^2 + \left(\frac{3}{\sqrt{x}}\right)^2\right) \\ &\ge (1+2+3)^2 = 36, \end{aligned}$$

   as desired. □

3. *Proof.* First we note that, from the AM–GM inequality,

   $$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{1}{ab} \ge \left(\frac{2}{a+b}\right)^2 = 4.$$

   Using Jensen's inequality with $f(x) = x^2$ gives us

   $$\begin{aligned} \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 &\ge \frac{1}{2}\left(a + \frac{1}{a} + b + \frac{1}{b}\right)^2 \\ &= \frac{1}{2}\left(1 + \frac{1}{a} + \frac{1}{b}\right)^2 \\ &\ge \frac{25}{2}, \end{aligned}$$

   proving the result. □

4. *Proof.* Suppose to the contrary such a sequence existed. The Cauchy–Schwarz inequality applied to the vectors (a_k) and (a_k^2) gives

   $$(a_1^3 + a_2^3 + \cdots)^2 \le (a_1^2 + a_2^2 + \cdots)(a_1^4 + a_2^4 + \cdots).$$

   From our assumption we deduce $9 \le 8$, a contradiction. □

5. *Proof.* Let's use the Cauchy–Schwarz inequality with the vectors (ka_k) and $(1/k)$. This gives

   $$(x_1 + x_2 + \cdots + x_n)^2 \le \left(\sum_{k=1}^{n} \frac{1}{k^2}\right)(x_1^2 + 4x_2^2 + 9x_3^2 + \cdots + n^2x_n^2).$$

   It is well–known that as $n \to \infty$ the sum $\sum(1/k^2)$ approaches $\pi^2/6$. Truncating the infinite series after n terms, we bound the finite sum above by this value, giving the result.

   Now assume that we replace $\pi^2/6$ with a universal constant C. Taking $x_k = 1/k^2$ gives

   $$\begin{aligned} \left(\sum_{k=1}^{n} \frac{1}{k^2}\right)^2 &= (x_1 + x_2 + \cdots + x_n)^2 \\ &\le C(x_1^2 + 4x_2^2 + 9x_3^2 + \cdots + n^2x_n^2) \\ &= C\sum_{k=1}^{n} \frac{1}{k^2}, \end{aligned}$$

   so that $1 + 1/4 + 1/9 + \cdots + 1/n^2 \le C$. Taking $n \to \infty$ gives $\pi^2/6 \le C$, as desired. □

6. *Proof.* Let $f : (0, 1) \to \mathbb{R}$ be given by $f(x) = x/(x^2 + 1)$. Then $f''(x) = 2x(x^2 - 3)/(x^2 + 1)^3 < 0$ since $0 < x < 1$. Thus f is concave and Jensen's inequality gives

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \leq 3f\left(\frac{a+b+c}{3}\right) = 3f\left(\frac{1}{3}\right) = \frac{9}{10}.$$

□

7. *Proof.* Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = x \ln x$. Then $f''(x) = 1/x > 0$, so f is convex. By Jensen's inequality we have

$$a \ln a + b \ln b + c \ln c \geq (a + b + c) \ln\left(\frac{a+b+c}{3}\right) \geq (a + b + c) \ln\left(\sqrt[3]{abc}\right),$$

where the last step follows from the AM–GM since the log function is increasing (that is, it preserves inequalities). Finally, rearranging gives

$$\ln(a^a b^b c^c) \geq \ln(abc)^{(a+b+c)/3}.$$

Again, since ln is increasing we can cancel it to obtain the result. □

8. *Proof.* Since ABC is an acute triangle, the angles A, B, C are each within $(0, \pi/2)$ and sum to π. On $(0, \pi/2)$ the function $\sec^2$ is convex, so Jensen's inequality gives

$$\sec^2 A + \sec^2 B + \sec^2 C \geq 3 \sec^2\left(\frac{A+B+C}{3}\right) = 3 \sec^2\left(\frac{\pi}{3}\right) = 12.$$

Similarly, the function $\ln \circ \sin$ is concave, so Jensen gives

$$\begin{aligned}
\ln(\sin A \sin B \sin C) &= \ln \sin A + \ln \sin B + \ln \sin C \\
&\leq 3 \ln \sin\left(\frac{A+B+C}{3}\right) \\
&= 3 \ln \sin\left(\frac{\pi}{3}\right) \\
&= \ln\left(\frac{3\sqrt{3}}{8}\right).
\end{aligned}$$

Since ln is an increasing function, we cancel to deduce the result.

There are many ways to prove Weitzenböck's inequality. One approach is as follows. Starting with a nonnegative sum of squares,

$$0 \leq (a - b)^2 + (b - c)^2 + (c - a)^2 = 2(a^2 + b^2 + c^2 - ab - bc - ca),$$

we obtain $a^2 + b^2 + c^2 \geq ab + bc + ca$. Next recall that the area of a triangle can be computed via

$$K = \frac{ab}{2} \sin C = \frac{bc}{2} \sin A = \frac{ca}{2} \sin B,$$

so we can write

$$a^2 + b^2 + c^2 \geq 2K(\csc A + \csc B + \csc C).$$

The cosecant function is convex on $(0, \pi)$, so Jensen gives

$$a^2 + b^2 + c^2 \geq 6K \csc\left(\frac{A+B+C}{3}\right) = 6K \csc(\pi/3) = 4K\sqrt{3},$$

as desired. □

9. *Proof.* Since the logarithm is concave, the general Jensen inequality gives

$$\ln(ab) = \frac{\ln a^p}{p} + \frac{\ln b^q}{q} \leq \ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right),$$

where we've used $1/p, 1/q$ as the weights. The exponential function preserves inequalities, so applying to the above gives Young's inequality.

Now we attack Hölder, beginning with a very powerful trick. Notice that the inequality holds true for $x_1, x_2, \ldots, x_n$ if and only if it holds for $cx_1, cx_2, \ldots, cx_n$ for any constant c (such an inequality is called homogeneous). Choosing c carefully, we can assume $\sum |x_k|^p = 1$. Similarly, we can renormalize the numbers (y_k) so that $\sum |y_k|^q = 1$. It now suffices to prove $\sum |x_k y_k| \le 1$. By Young's inequality we have

$$|x_k y_k| \le \frac{|x_k|^p}{p} + \frac{|y_k|^q}{q}$$

for each k. Summing over k gives

$$\sum_{k=1}^{n} |x_k y_k| \le \frac{1}{p} \left(\sum_{k=1}^{n} |x_k|^p \right) + \frac{1}{q} \left(\sum_{k=1}^{n} |y_k|^q \right) = \frac{1}{p} + \frac{1}{q} = 1,$$

proving the result. □

10. *First Proof.* Rearranging the result, we see it's sufficient to prove

$$\sum_{k=1}^{n} p_k \ln\left(\frac{q_k}{p_k}\right) \le 0.$$

First we claim that whenever $x > 0$, we have $\ln x \le x - 1$. One way to do this is with calculus; define $f : (0, \infty) \to \mathbb{R}$ with $f(x) = x - 1 - \ln x$ and notice that $f(1) = 0$. When $x > 1$ we have $f'(x) = 1 - 1/x > 0$, so $f(x) > 0$ for all such x. When $0 < x < 1$ we have $f'(x) = 1 - 1/x < 0$, so $f(x) > 0$ for those x as well. This proves the claim.

With the claim in hand we have

$$\sum_{k=1}^{n} p_k \ln\left(\frac{q_k}{p_k}\right) \le \sum_{k=1}^{n} p_k \left(\frac{q_k}{p_k} - 1\right) = \sum_{k=1}^{n} (q_k - p_k) = 0,$$

and we're done. □

Second Proof. We can use Jensen instead. We're already seen that $f(x) = x \ln x$ is convex on $(0, \infty)$. Using (q_k) as weights in Jensen's inequality gives

$$\sum_{k=1}^{n} p_k \ln\left(\frac{p_k}{q_k}\right) = \sum_{k=1}^{n} q_k f\left(\frac{p_k}{q_k}\right) \ge f\left(\sum_{k=1}^{n} q_k \cdot \frac{p_k}{q_k}\right) = f(1) = 0,$$

as desired. □