Math 8 Homework 9 Solutions

1 Arithmetic and Algebra of Complex Numbers

(a) (a) $(1+i)^{100} = (\sqrt{2}e^{i\pi/4})^{100} = 2^{50}$.

(b) $(1+e^{i\theta})^n = e^{in\theta/2}(2^n \cos^n \theta/2)$.

(c) $\exp(e^{i\theta}) = \exp(\cos\theta + i\sin\theta) = e^{\cos\theta}e^{i\sin\theta} = e^{\cos\theta}(\cos(\sin\theta) + i\sin(\sin\theta))$

(b) (i) The set of all points 4 units away from 1 is a circle of radius 4 centered at 1.

(ii) The set of points equidistant from i and 4 is the perpendicular bisector of the line segment joining i and 4.

(iii) Given $z \in \mathbb{C}$, note that $|z|$ is distance to 0 while $\mathrm{Re}(z+2)$ is the distance to the vertical line $y = -2$. The set for which these are equal is a parabola.

(c) (i) $z^6 - 2z^3 + 2 = 0$ gives $(z^3-1)^2 = -1$, so that $z^3 = 1 \pm i$. Converting to polar, $z^3 = \sqrt{2}e^{\pm i\pi/4}$ and hence $z = 2^{1/6}\exp(2ik\pi/3 \pm i\pi/12)$, where $k \in \{0,1,2\}$.

(ii) $(z+1)^5 = z^5$ becomes $(1+1/z)^5 = 1$ upon division. Hence $z = 1/(\exp(2ik\pi/5)-1)$ with $k \in \{0,1,2,3,4\}$. We exclude $k=0$ since it leads to division by 0.

(iii) $e^z = 1+i$ becomes $e^z = \sqrt{2}e^{i\pi/4}$ when we rewrite in polar. The real part of $z = x+iy$ must satisfy $e^x = \sqrt{2}$, so that $x = \ln(2)/2$. The imaginary part must satisfy $y = \pi/4 + 2\pi k$ for some $k \in \mathbb{Z}$. Thus $z = \ln\sqrt{2} + i\pi/4 + 2ki\pi$ with $k \in \mathbb{Z}$.

(iv) $z^4 = 5(z-1)(z^2-z+1)$ becomes $z^4 - 5z^3 + 10z^2 - 10z + 5 = 0$ after rearranging. This is suspiciously familiar, so multiplying by z and subtracting 1 gives

$$(z-1)^5 = -1.$$

That is, $z = 1 + e^{(2k+1)i\pi/5}$ with $k \in \{0,1,2,3,4\}$. However, $k=0$ leads to $z=0$ which doesn't satisfy the original equation. So there are only 4 solutions (as can be expected from a quartic equation).

(d) *Proof.* Rewrite $\cos(n\theta) = \mathrm{Re}(e^{in\theta})$ and evaluate the geometric series:

$$\sum_{n=0}^{\infty}\frac{\cos(n\theta)}{2^n} = \mathrm{Re}\sum_{n=0}^{\infty}\left(\frac{e^{i\theta}}{2}\right)^n = \mathrm{Re}\frac{1}{1-e^{i\theta}/2}.$$

The rest is algebra.

$$\sum_{n=0}^{\infty}\frac{\cos(n\theta)}{2^n} = \mathrm{Re}\frac{1}{1-e^{i\theta}/2} = \mathrm{Re}\frac{2}{2-e^{i\theta}} = \mathrm{Re}\frac{2(2-e^{-i\theta})}{(2-e^{i\theta})(2-e^{-i\theta})} = \mathrm{Re}\frac{4-2e^{-i\theta}}{5-2(e^{i\theta}+e^{-i\theta})}.$$

The denominator contains an expression for $2\cos\theta$:

$$\sum_{n=0}^{\infty}\frac{\cos(n\theta)}{2^n} = \mathrm{Re}\frac{4-2\cos\theta+2i\sin\theta}{5-4\cos\theta} = \frac{4-2\cos\theta}{5-4\cos\theta},$$

and we're done. □

(e) *Proof.* TYPO ALERT: I should've written *nonconstant* in the problem statement. Let $a \in \mathbb{C}$ be given. Then $p(z) - a$ is a nonconstant polynomial, and by the fundamental theorem of algebra, there exists $z \in \mathbb{C}$ so that $p(z) - a = 0$. Hence $p(z) = a$ and p is surjective. □

(f) *Proof.* Define the function

$$f(z) = \frac{z-p}{1-\overline{p}z}.$$

Clearly $f(p) = 0$. Furthermore, the denominator of f never vanishes; for $z \in \mathbb{D}$ we have $|\overline{p}z| < 1$. Finally, note that as $|z| \to 1$ we have $\overline{z} \to 1/z$ so that

$$|f(z)| = \left|\frac{z-a}{1-\overline{z}a}\right| \to \left|\frac{z-a}{1-a/z}\right| = |z| = 1,$$

as desired. □

2 Problem Solving with Complex Numbers

(a) *Proof.* Let $\omega = \exp(2i\pi/n)$. The product of the lengths we want is

$$P = \prod_{k=1}^{n-1} |1 - \omega^k|.$$

Let $f(z) = 1 + z + z^2 + \cdots + z^{n-1}$, whose roots are $\omega, \omega^2, \ldots, \omega^{n-1}$. Then $g(z) = f(1-z)$ is a polynomial with roots $1-\omega, 1-\omega^2, \ldots, 1-\omega^{n-1}$. The product of the roots of a polynomial g with degree $n-1$ is $(-1)^{n-1}g(0)$, so we have

$$P = \left| \prod_{k=1}^{n-1} (1 - \omega^k) \right| = |(-1)^{n-1} g(0)| = |f(1)| = n.$$

Now we want to compute

$$S = \sum_{k=1}^{n-1} |1 - \omega^k|^2 = \sum_{k=1}^{n-1} (1 - \omega^k)(1 - \omega^{n-k}) = \sum_{k=1}^{n-1} (1 - \omega^k - \omega^{n-k} + \omega^n).$$

The sum $\omega + \omega^2 + \cdots + \omega^{n-1} = -1$. Using this and the fact that $\omega^n = 1$,

$$S = 2(n-1) - (-1) - (-1) = 2n,$$

and we're done. □

(b) *Proof.* Let a, b, m, n be positive integers. Then we have

$$(a^2 + b^2)(m^2 + n^2) = |a + bi|^2 |m + ni|^2 = |(am - bn) + i(an + bm)|^2 = (am - bn)^2 + (an + bm)^2.$$

Hence the product of two sums of 2 squares is itself a sum of 2 squares. □

(c) *Proof.* Set $z \mapsto \omega z$ to get

$$f(\omega z) + f(\omega^2 z) = \exp(\omega z). \tag{1}$$

In this equation again set $z \mapsto \omega z$ to find

$$f(\omega^2 z) + f(\omega^3 z) = \exp(\omega^2 z).$$

The second term above simplifies since $\omega^3 = 1$, so we find

$$f(\omega^2 z) + f(z) = \exp(\omega^2 z). \tag{2}$$

If we take the given identity

$$f(z) + f(\omega z) = \exp(z),$$

subtracting (1), adding (2), and dividing by 3 gives

$$f(z) = \frac{\exp(z) - \exp(\omega z) + \exp(\omega^2 z)}{3}.$$

Proving uniqueness of f is immediate. Starting from the identity from f we derived exactly what f must be, so there can be no other f. □

(d) *Proof.* We begin with a geometric series. When $z \in \mathbb{D}$ the series

$$z^a + z^{a+d} + z^{a+2d} + \cdots = z^a(1 + z^d + (z^d)^2 + \cdots) = \frac{z^a}{1 - z^d}$$

is a convergent geometric series.

Now suppose that we find integers a_k, d_k so that each d_k is distinct and

$$\frac{z}{1-z} = \frac{z^{a_1}}{1 - z^{d_1}} + \frac{z^{a_2}}{1 - z^{d_2}} + \cdots + \frac{z^{a_n}}{1 - z^{d_n}}$$

for all $z \in \mathbb{D}$. Without loss of generality let d_1 be the largest such d_k. Multiplying gives

$$\frac{z(1-z^{d_1})}{1-z} = z^{a_1} + (1-z^{d_1})\left(\frac{z^{a_2}}{1-z^{d_2}} + \cdots + \frac{z^{a_n}}{1-z^{d_n}}\right) \tag{3}$$

Now we have to keep $z \in \mathbb{D}$, but we can take the limit $z \to e^{2i\pi/d_1}$ while z stays within the disk. Assuming $d_1 \neq 1$ we deduce

$$0 = e^{2ia_1\pi/d_1},$$

which is impossible. Hence the largest d_k is 1 and as all d_k were distinct, there are no others. That is, $n = 1$. Returning to equation (3) we see that $z = z^{a_1}$, so $a_1 = 1$. This proves the second result.

Finally, assume we can decompose $\mathbb{N}$ into a disjoint collection of arithemtic progressions:

$$\mathbb{N} = \{a_1, a_1 + d_1, a_1 + 2d_1, \ldots\} \cup \cdots \cup \{a_n, a_n + d_n, a_n + 2d_n, \ldots\},$$

with each d_k distinct. We can use this partition to write

$$z + z^2 + z^3 + \cdots = (z^{a_1} + z^{a_1+d_1} + \cdots) + \cdots + (z^{a_n} + z^{a_n+d_n} + \cdots)$$

for all $z \in \mathbb{D}$. Summing these series gives

$$\frac{z}{1-z} = \frac{z^{a_1}}{1-z^{d_1}} + \frac{z^{a_2}}{1-z^{d_2}} + \cdots + \frac{z^{a_n}}{1-z^{d_n}}.$$

By the above reasoning, $n = a_1 = d_1 = 1$. That is, our partition of $\mathbb{N}$ must have been the trivial one. $\square$