Math 8, Summer 2012
Practice Exam 2

Short Ans.	
1	
2	
3	
Total	

Name______________________________

Perm No.______________________________

Directions:

1. Each problem is graded out of 4 points.
2. Each short answer question is worth 1 point.
3. You're only allowed a writing instrument and your wits.
4. Proofs should be clean, to the point, and written in proper English sentences.

Short Answer

1. Given $n \in \mathbb{N}$ evaluate $\displaystyle\sum_{k=0}^{n} \frac{1}{(n-k)!(n+k)!}$.

2. Billy claims to have an a data compression algorithm that takes any 100–bit string of 0s and 1s and reduce its size to 50 bits. Explain to Billy why he must be destroying information; that is, there must be two different strings that get compressed into the same result.

3. In how many ways can we divide 6 students into 2 nonempty groups? The groups are functionally identical, except for the people in them.

4. A relation $\simeq$ is called injective if whenever x, y, z in the underlying set satisfy $x \simeq y$ and $z \simeq y$, it must follow that $x = z$. Give an example of such a relation.

5. Precisely define what it means for a relation $\sim$ on a set S to be antisymmetric.

6. Given an equivalence relation R on a set S, precisely define S/R.

7. Give a combinatorial defintion of $\binom{n}{k}$ (no formulas)

8. How many two–element subsets $\{a, b\}$ of $\{1, 2, \ldots, 50\}$ satisfy $|a - b| = 5$?

Problems

1. Let S be a nonempty set and $\text{Aut}(S)$ denote the set of all bijective functions $S \to S$. Given functions $f, g \in \text{Aut}(S)$ define $f \sim g$ if and only if there is $h \in \text{Aut}(S)$ so that $f \circ h = h \circ g$. Prove that $\sim$ is an equivalence relation on $\text{Aut}(S)$.

   Remark: In group theory, when $f \sim g$ we say the two functions are 'conjugate'.

2. Let $F(n,k)$ denote the number of surjective functions $\{1,2,\ldots,n\} \to \{1,2,\ldots,k\}$. There is no simple formula for computing $F(n,k)$ in general, but we have a recurrence relation:

$$F(n+1,k) = k \cdot F(n,k-1) + k \cdot F(n,k).$$

First figure out $F(n,n)$ and $F(2,1)$ directly. Then use the recurrence relation and induction to prove for all integers $n \geq 2$,

$$F(n,n-1) = \binom{n}{2} \cdot (n-1)!$$

Remark: As an extra challenge, use combinatorial reasoning to prove $F(n,k) = k!S(n,k)$, where S denotes the Stirling number of the second kind.

3. Let n be a positive integer. Given an integer $k \leq n$ define the $D_n(k)$ to be the number of ways to permute n students so that **exactly** k objects end up in their starting positions. Use combinatorial reasoning to prove that

$$\sum_{k=0}^{n} k \cdot D_n(k) = n!$$