Math 8, Summer 2012
Practice Final

Short Ans.	
1	
2	
3	
Total	

Name______________________________

Perm No.___________________________

Directions:

1. Each problem is graded out of 4 points.
2. Each short answer question is worth 1 point.
3. You're only allowed a writing instrument and your wits.
4. Proofs should be clean, to the point, and written in proper English sentences.

Short Answer

1. Precisely define what it means for a set A to be uncountable.

2. Precisely define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be convex.

3. Geometrically describe the set of all complex z so that $z^4 \operatorname{Re}(z) = \overline{z}^4 \operatorname{Im}(z)$.

4. Let $A_1, A_2, A_3, \ldots$ be a collection of countable sets. Which of these need not be countable?
   (a) $A_1 - A_2$
   (b) $\displaystyle\bigcup_{n=1}^{\infty} A_n$
   (c) $A_1 \times A_2 \times A_3 \times \cdots$
   (d) $\displaystyle\bigcup_{n=1}^{\infty} (A_1 \times A_2 \times \cdots \times A_n)$
   (e) None of the above

5. How many distinct $z \in \mathbb{C}$ satisfy $e^z = 16$?

   (a) 0
   (b) 1
   (c) 2
   (d) 4
   (e) Infinitely many

6. Find the minimum value of $(\sin x \sec x \tan x) + (\csc x \cos x \cot x)$ for $0 < x < \pi/2$.

7. What is the real part of $(1 + i\sqrt{3})^{2012}$?

8. Does there exist a set S so that

$$|\mathbb{N}| < |S| < |\mathcal{P}(\mathbb{N})|$$

   (a) Yes, $\mathbb{R}$ works
   (b) Yes, $\mathbb{Q}$ works
   (c) Yes, but the set is nearly indescribable
   (d) No
   (e) How the hell should I know!?

Problems

1. A real number α is called algebraic if there exists a polynomial $p(x)$ with integer coefficients so that $p(\alpha) = 0$. A real number that is not algebraic is called transcendental. For centuries mathematicians were unsure whether transcendental numbers exist. Prove that transcendental numbers exist by following this outline (due to G. Cantor):

   (a) Given $n \in \mathbb{N}$ let A_n denote the set of all algebraic numbers that satisfy a polynomial equation
   $$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0.$$
   Let B_{nm} denote those elements of A_n that satisfy a polynomial equation as above, but with each $|a_k| \le m$. Prove that each B_{nm} is finite.

   (b) Prove that each A_n is countable.

   (c) Let A denote the set of all algebraic numbers. Prove that A is countable.

   (d) Let T denote the set of all transcendental numbers. Prove that T is nonempty.

2. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of real numbers, not all zero. Prove Carlson's second inequality:

$$(a_1 + a_2 + a_3 + \cdots)^4 \leq \pi^2(a_1^2 + a_2^2 + a_3^2 + \cdots)(a_1^2 + 4a_2^2 + 9a_3^2 + \cdots),$$

with equality if and only if the right side is infinite. Proceed by following this outline (due to G.H. Hardy).

(a) First address trivial cases:
- $\sum_k k^2 a_k^2 = \infty$
- $\sum_k a_k^2 = \infty$

(b) Let $\alpha, \beta > 0$ be unspecifed parameters to be chosen later. Use Cauchy-Schwarz on the vectors $(1/\sqrt{\alpha + \beta k^2})$ and $(a_k \sqrt{\alpha + \beta k^2})$.

(c) Prove that

$$\sum_{k=1}^{\infty} \frac{1}{\alpha + \beta k^2} < \int_0^{\infty} \frac{dx}{\alpha + \beta x^2}.$$

(d) Evaluate the aforementioned integral in terms of α and β.

(e) Finally, choose

$$\alpha = \left(\sum_{k=1}^{\infty} k^2 a_k^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} a_k^2 \right)^{-1/2}$$

and $\beta = 1/\alpha$. Simplify to deduce the result.

3. Given $z, \zeta \in \mathbb{C}$ such that $|z| < 1$ and $|\zeta| = 1$, define the Poisson kernel (of the disk) as

$$P(z, \zeta) = \frac{1 - |z|^2}{2\pi |z - \zeta|^2}.$$

Prove whenever $|z| < 1$ that

$$\int_0^{2\pi} P(z, e^{it})\, dt = 1$$

by following this outline (in this version due to T. Ransford).

(a) Let $0 \le r < 1$ and $\theta, t \in [0, 2\pi)$. Take the series

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta - t)}$$

and rewrite it as two infinite series—one from $n = 0$ to $n = \infty$ and one from $n = -1$ to $n = -\infty$.

(b) Sum the two (now geometric) series you found to show that

$$P(re^{i\theta}, e^{it}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta - t)}.$$

(c) Swap sum and integral (assume this is valid—it is, but we don't have the background to prove it) to finish the proof.

Remark: This shows the Poisson kernel acts as a probability ditribution on the unit circle—a fact that allows a stochastic interpretation of harmonic function theory.