Math 8, Summer 2012
Practice Exam 1 Solutions

Short Answer

1. (A logic puzzle) Among the following three statements, indicate which one(s) is/are true.

    - All three of these statements are false.
    - Exactly two of these statements are false.
    - Exactly one of these statements is false.

    At most one statement can be true, as they each contradict the others. If all were false, the first would in fact be true. Hence exactly one is true—the middle one.

2. Let A be a set. Give a precise definition of $\mathcal{P}(A)$.

    $\mathcal{P}(A) = \{T : T \subseteq A\}$

3. Give a precise definition of what it means for $f : A \to B$ to be bijective.

    For every $y \in B$ there is a unique $x \in A$ so that $f(x) = y$.

4. Let P and Q be statements. Under what circumstances is (P or Q) false?

    (P or Q) is false if and only if P and Q are both false.

5. Give a precise definition of $\varnothing$.

   For all x we have $x \notin \varnothing$.

6. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is surjective but not injective.

   Consider $f : x \mapsto x \sin x$. Clearly f is not injective (it has zeroes at every integer multiple of π). Since f takes arbitrarily large positive and negative values—in particular $f(\pi \cdot (n + 1/2)) = \pi \cdot (n + 1/2)(-1)^n$ for any integer n—the intermediate value theorem implies f is surjective.

7. Using the axiom of choice we can reorder $\mathbb{R}$ so that every subset of $\mathbb{R}$ has a least element. We can define a function that produces the least element from a given subset. Specify the domain and codomain of this function.

   This function maps $\mathcal{P}(\mathbb{R}) \to \mathbb{R}$.

8. Give a precise definition of what it means for an integer n to be even.

   There exists an integer m so that $n = 2m$.

Problems

1. Prove that $\log_2(3)$ is irrational.

   Proof. Assume to the contrary that $\log_2(3) = m/n$ for some integer m and n. Since $\log_2(3) > 0$ we have $m, n > 0$. From here we find $2^{m/n} = 3$, or rather $2^m = 3^n$. Since $m > 0$, this implies 3^n is even, a contradiction. $\square$

2. Prove that

$$\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{1, 2, 3, \ldots, m\} = \mathbb{N}$$

Proof. For simplicity denote the left hand side as S. Clearly any element of S is a positive integer, so $S \subseteq \mathbb{N}$. Given any $n \in \mathbb{N}$ and $m \geq n$ note that $n \in \{1, 2, \ldots, m\}$. Thus

$$n \in \bigcap_{m=n}^{\infty} \{1, 2, \ldots, m\}.$$

This implies $n \in S$. We conclude that $\mathbb{N} \subseteq S$ and thus $\mathbb{N} = S$. $\square$

3. Let $f : A \to B$ be a function. Prove that f is injective if and only if for any sets $S, T \subseteq A$ we have $f(S \cap T) = f(S) \cap f(T)$.

Proof. ($\Rightarrow$) Assume that f is injective and consider arbitrary sets $S, T \subseteq A$. Given $y \in f(S \cap T)$ find $x \in S \cap T$ so that $f(x) = y$. Since $x \in S$ this implies $y \in f(S)$. Similarly $x \in T$ and hence $y \in f(T)$. Therefore $y \in f(S) \cap f(T)$, so $f(S \cap T) \subseteq f(S) \cap f(T)$.
Now suppose $c \in f(S) \cap f(T)$. Since $c \in f(S)$ there is $a \in S$ so that $f(a) = c$. Similarly there is $b \in T$ so that $f(b) = c$. The injectivity of f however forces $a = b$. Hence $a \in S \cap T$ and $f(a) = c$, so we deduce $c \in f(S \cap T)$. This implies that $f(S) \cap f(T)$ is contained within $f(S \cap T)$, and thus the sets are equal.

($\Leftarrow$) Assume that f is not injective. There are $x, y \in A$ so that $f(x) = f(y)$ although $x \neq y$. Define $S = \{x\}$ and $T = \{y\}$. Then we have $f(S) \cap f(T) = f(S) \neq \varnothing$, although $f(S \cap T) = f(\varnothing) = \varnothing$. $\square$