Math 8, Summer 2012
Practice Exam 2 Solutions

Short Answer

1. Given $n \in \mathbb{N}$ evaluate $\displaystyle\sum_{k=0}^{n} \frac{1}{(n-k)!(n+k)!}$.

   Call the answer A. Make the terms look like binomial coefficients:

   $$A = \frac{1}{(2n)!}\left(\binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{n}\right)$$

   Use symmetry:

   $$\begin{aligned} A + A &= \frac{1}{(2n)!}\left(\binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{n}\right) + \frac{1}{(2n)!}\left(\binom{2n}{n} + \binom{2n}{n+1} + \cdots + \binom{2n}{2n}\right) \\ &= \frac{1}{(2n)!}\left(2^{2n} + \binom{2n}{n}\right). \end{aligned}$$

   The answer is half of this number.

2. Billy claims to have an a data compression algorithm that takes any 100–bit string of 0s and 1s and reduce its size to 50 bits. Explain to Billy why he must be destroying information; that is, there must be two different strings that get compressed into the same result.

   The pigeonhole principle guarantees that if 2^{100} 100–bit binary strings are compressed into 2^{50} 50–bit strings that some 2 distinct strings will be compressed into the same result. If we can't undo the procedure, we've lost information.

3. In how many ways can we divide 6 students into 2 nonempty groups? The groups are functionally identical, except for the people in them.

   We want the Stirling number $S(6, 2)$. We can divide the students into 2 groups in $2^6 = 64$ ways. However, 2 of these ways produce an empty group, leaving only 62 ways. We've tacitly assumed the groups are distinct, so dividing by 2 gives 31 ways.

4. A relation $\simeq$ is called injective if whenever x, y, z in the underlying set satisfy $x \simeq y$ and $z \simeq y$, it must follow that $x = z$. Give an example of such a relation.

   A lazy answer is standard equality, $=$, on any set. Another lazy example is the empty relation on any set. A better answer is $x \simeq y$ if and only if $x, y \in \mathbb{R}$ and $x = e^y$.

5. Precisely define what it means for a relation $\sim$ on a set S to be antisymmetric.

For all $x, y \in S$ we have $x \sim y$ and $y \sim x$ implies $x = y$.

6. Given an equivalence relation R on a set S, precisely define S/R.

$S/R = \{[x] : x \in S\} = \{\{y \in S : y \sim x\} : x \in S\}$, the set of equivalence classes.

7. Give a combinatorial defintion of $\binom{n}{k}$ (no formulas)

$\binom{n}{k}$ is the number of k–element subsets of an n–element set.

8. How many two–element subsets $\{a, b\}$ of $\{1, 2, \ldots, 50\}$ satisfy $|a - b| = 5$?

The only subsets are $\{1, 6\}, \{2, 7\}, \{3, 8\}, \ldots, \{45, 50\}$, so there are 45 such sets.

Problems

1. Let S be a nonempty set and $\mathrm{Aut}(S)$ denote the set of all bijective functions $S \to S$. Given functions $f, g \in \mathrm{Aut}(S)$ define $f \sim g$ if and only if there is $h \in \mathrm{Aut}(S)$ so that $f \circ h = h \circ g$. Prove that $\sim$ is an equivalence relation on $\mathrm{Aut}(S)$.

   Remark: In group theory, when $f \sim g$ we say the two functions are 'conjugate'.

   Proof. Let $i : S \to S$ denote the identity map and note that $i \in \mathrm{Aut}(S)$. Given $f \in \mathrm{Aut}(S)$ we have $i \circ f = f \circ i$, so $f \sim f$. This shows $\sim$ is reflexive.

   Given $f, g \in \mathrm{Aut}(S)$ so that $f \sim g$, find $h \in \mathrm{Aut}(S)$ so that $f \circ h = h \circ g$. Since h is bijective h^{-1} exists and is also bijective. We find that

$$h^{-1} \circ f = h^{-1} \circ f \circ h \circ h^{-1} = h^{-1} \circ h \circ g \circ h^{-1} = g \circ h^{-1},$$

   so we conclude $g \sim f$. This shows $\sim$ is symmetric.

   Given $f, g, \varphi \in \mathrm{Aut}(S)$ so that $f \sim g$ and $g \sim \varphi$, find $h_1, h_2 \in \mathrm{Aut}(S)$ so that $f \circ h_1 = h_1 \circ g$ and $g \circ h_2 = h_2 \circ \varphi$. Then we have

$$f \circ (h_1 \circ h_2) = h_1 \circ g \circ h_2 = (h_1 \circ h_2) \circ \varphi.$$

   Since the composition of bijections is bijective, $h_\circ h_2 \in \mathrm{Aut}(S)$. This shows $f \sim \varphi$ and $\sim$ is transitive. □

2. Let $F(n,k)$ denote the number of surjective functions $\{1,2,\ldots,n\} \to \{1,2,\ldots,k\}$. There is no simple formula for computing $F(n,k)$ in general, but we have a recurrence relation:

$$F(n+1,k) = k \cdot F(n,k-1) + k \cdot F(n,k).$$

First figure out $F(n,n)$ and $F(2,1)$ directly. Then use the recurrence relation and induction to prove for all integers $n \geq 2$,

$$F(n,n-1) = \binom{n}{2} \cdot (n-1)!$$

Remark: As an extra challenge, use combinatorial reasoning to prove $F(n,k) = k!S(n,k)$, where S denotes the Stirling number of the second kind.

Proof. First note that any surjection from $\{1,2,\ldots,n\}$ to itself is a permutation. That is, $F(n,n) = n!$. Furthermore, $F(2,1) = 1$ since there is only one function $\{1,2\} \to \{1\}$, and it is surjective. Now we proceed with the induction. The base case we've already addressed, so assume that

$$F(n,n-1) = \binom{n}{2} \cdot (n-1)!$$

for some integer $n \geq 2$. Using the recurrence relation we have that

$$\begin{aligned} F(n+1,n) &= n \cdot F(n,n-1) + n \cdot F(n,n) \\ &= n\binom{n}{2} \cdot (n-1)! + n \cdot n! \\ &= \frac{n(n-1)}{2} \cdot n! + n \cdot n! \\ &= \binom{n+1}{2} \cdot n!, \end{aligned}$$

completing the induction. □

Remark: To see the relationship to Stirling numbers, consider the following. To build a surjection from an n–element set to a k–element set, first divide the n elements into k nonempty indistinguishable groups in $S(n,k)$ ways. Then assign each group to a different element of $\{1,2,\ldots,k\}$ in $k!$ ways.

3. Let n be a positive integer. Given an integer $k \leq n$ define the $D_n(k)$ to be the number of ways to permute n students so that **exactly** k objects end up in their starting positions. Prove that

$$\sum_{k=0}^{n} k \cdot D_n(k) = n!$$

Proof. Imagine n students seated in a row. Some number of them rearrange themselves in the chairs; choose one student who doesn't move and hand her a shiny new toy. There are $D_n(k)$ ways to arrange students wherein k students don't move. After that pick a student to receive a toy in k ways. The total number of ways to do all of this is $\sum_k kD_n(k)$.

Alternatively, pick a student to receive a toy in n ways. Then permute the other students arbitrarily in $(n-1)!$ ways. The total number of ways this can occur is $n \cdot (n-1)! = n!$. Having counted the situation in two ways, the two counts must be equal. □