Math 8, Summer 2012
Practice Final Solutions

Short Answer

1. Precisely define what it means for a set A to be uncountable.

   A few equivalent answers:

   - There does not exist an injection $A \to \mathbb{N}$
   - There does not exist a surjection $\mathbb{N} \to A$
   - A is infinite but there is no bijection $A \to \mathbb{N}$

2. Precisely define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be convex.

   For all $x, y \in \mathbb{R}$ and $0 < t < 1$ we have $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.

3. Geometrically describe the set of all complex z so that $z^4 \operatorname{Re}(z) = \overline{z}^4 \operatorname{Im}(z)$.

   Let $z = re^{i\theta}$ to get either $r = 0$ or $e^{8i\theta} = \tan\theta$. In the latter case, $e^{8i\theta} \in \mathbb{R}$, so $8\theta = n\pi$ for some $n \in \mathbb{Z}$. Also, $\cos(n\pi) = \tan(n\pi/8)$; checking $n = 0, 1, \ldots, 15$ gives only $n = 2, 10$. From $z = 0$ or $z = re^{i\pi/4}$ or $z = re^{5i\pi/4}$ for arbitrary r, we see the solution is the line $y = x$ through the origin.

4. Let $A_1, A_2, A_3, \ldots$ be a collection of countable sets. Which of these need not be countable?

   (a) $A_1 - A_2$

   (b) $\displaystyle\bigcup_{n=1}^{\infty} A_n$

   (c) $A_1 \times A_2 \times A_3 \times \cdots$

   (d) $\displaystyle\bigcup_{n=1}^{\infty} (A_1 \times A_2 \times \cdots \times A_n)$

   (e) None of the above

   (c) is the answer. For example, there is a bijection $\mathcal{P}(\mathbb{N}) \to \{0,1\} \times \{0,1\} \times \{0,1\} \times \cdots$

5. How many distinct $z \in \mathbb{C}$ satisfy $e^z = 16$?

   (a) 0
   (b) 1
   (c) 2
   (d) 4
   (e) Infinitely many

   (e) is the answer. There is at least one answer since $e^{\ln 16} = 16$; there are infinitely many because $e^{z+2in\pi} = e^z$ for any integer n.

6. Find the minimum value of $(\sin x \sec x \tan x) + (\csc x \cos x \cot x)$ for $0 < x < \pi/2$.

   Let $f(x) = \sin x \sec x \tan x$. When $0 < x < \pi/2$ we see that $f(x) > 0$, so the AM–GM gives

$$(\sin x \sec x \tan x) + (\csc x \cos x \cot x) = f(x) + \frac{1}{f(x)} \geq 2$$

   with equality if and only if $f(x) = 1$ for some x. Since $f(0) = 0$ and $f(x) \to \infty$ as $x \to \pi/2$, the intermediate value theorem guarantees that $f(x) = 1$ at some x, so the answer is 2.

7. What is the real part of $(1 + i\sqrt{3})^{2012}$?

   $(1 + i\sqrt{3})^{2012} = 2^{2012}(1/2 + i\sqrt{3}/2)^{2012} = 2^{2012}e^{2012i\pi/3} = 2^{2012}e^{2i\pi/3} = 2^{2012}(-1/2 + i\sqrt{3}/2)$. The answer is 2^{2011}.

8. Does there exist a set S so that

$$|\mathbb{N}| < |S| < |\mathcal{P}(\mathbb{N})|$$

   (a) Yes, $\mathbb{R}$ works
   (b) Yes, $\mathbb{Q}$ works
   (c) Yes, but the set is nearly indescribable
   (d) No
   (e) How the hell should I know!?

   The answer is (e). Really. The so–called Continuum Hypothesis is independent of the typical (ZF) axioms of set theory. The question is undecidable in that framework and can be taken to be either true or false without contradiction.

Problems

1. A real number α is called algebraic if there exists a polynomial $p(x)$ with integer coefficients so that $p(\alpha) = 0$. Let A be the set of real algebraic numbers. A real number that is not algebraic is called transcendental. For centuries mathematicians were unsure whether transcendental numbers exist. Prove that transcendental numbers exist.

   Proof. Let $n, m \in \mathbb{N}$. Let A_n be the set of $x \in A$ which solve a polynomial equation $p(x) = 0$ with the degree of p at most n. Let B_{nm} be the set of $x \in A_n$ which satisfy a polynomial equation $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with each $|a_k| \leq m$. If we consider such a polynomial, then there are $2m + 1$ choices for each coefficient, giving $(2m+1)^{n+1}$ polynomials. Each has at most n roots, so there are no more than $n(2m+1)^{n+1}$ elements in B_{nm}. Hence B_{nm} is finite.

   Any element $x \in A_n$ solves some polynomial equation $p(x) = a_0 + a_1 x + \cdots + a_n x^n = 0$. If we let $M = \max\{|a_0|, |a_1|, \ldots, |a_n|\}$, then $x \in B_{nM}$. That is,

   $$A_n = \bigcup_{m=1}^{\infty} B_{nm}.$$

   Since A_n is a countable union of finite sets, A_n is countable.

   Any element $x \in A$ solves a polynomial equation $p(x) = 0$. If the degree of p is n, then $x \in A_n$. This shows

   $$A = \bigcup_{n=1}^{\infty} A_n.$$

   Since A is a countable union of countable sets, A is countable.

   Suppose there were no transcendental numbers. Then $\mathbb{R} = A$ is countable, a contradiction. Hence transcendental numbers exist (and are in fact far more numerous than algebraic ones). $\square$

2. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of real numbers, not all zero. Prove Carlson's second inequality:

$$(a_1 + a_2 + a_3 + \cdots)^4 \leq \pi^2 (a_1^2 + a_2^2 + a_3^2 + \cdots)(a_1^2 + 4a_2^2 + 9a_3^2 + \cdots),$$

with equality if and only if the right side is infinite.

Proof. If either sum on the right side is infinite, we have $(\sum a_k)^4 \leq \infty$, which is true. So now assume both sums on the right are finite. Let $\alpha, \beta > 0$ be unspecified parameters to be chosen later. Using Cauchy–Schwarz on the vectors $(1/\sqrt{\alpha + \beta k^2})$ and $(a_k \sqrt{\alpha + \beta k^2})$ gives

$$\left(\sum_{k=1}^{\infty} a_k \right)^2 \leq \left(\sum_{k=1}^{\infty} \frac{1}{\alpha + \beta k^2} \right) \left(\alpha \sum_{k=1}^{\infty} a_k^2 + \beta \sum_{k=1}^{\infty} k^2 a_k^2 \right) \tag{$\star$}$$

Note that, since $1/(\alpha + \beta x^2)$ is a decreasing function of x on $(0, \infty)$,

$$\int_0^{\infty} \frac{dx}{\alpha + \beta x^2} = \sum_{k=1}^{\infty} \int_{k-1}^{k} \frac{dx}{\alpha + \beta x^2} > \sum_{k=1}^{\infty} [k - (k-1)] \frac{1}{\alpha + \beta k^2} = \sum_{k=1}^{\infty} \frac{1}{\alpha + \beta k^2}$$

This integral is easy to evaluate analytically using arctangent. This gives

$$\sum_{k=1}^{\infty} \frac{1}{\alpha + \beta k^2} < \frac{\pi}{2\sqrt{\alpha\beta}}.$$

Inserting this into equation ($\star$) gives

$$\left(\sum_{k=1}^{\infty} a_k \right)^2 < \frac{\pi}{2\sqrt{\alpha\beta}} \left(\alpha \sum_{k=1}^{\infty} a_k^2 + \beta \sum_{k=1}^{\infty} k^2 a_k^2 \right)$$

Since this is true for all positive α, β we can choose

$$\alpha = \left(\sum_{k=1}^{\infty} k^2 a_k^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} a_k^2 \right)^{-1/2}$$

and $\beta = 1/\alpha$ to get

$$\left(\sum_{k=1}^{\infty} a_k \right)^2 < \pi \left(\sum_{k=1}^{\infty} a_k^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} k^2 a_k^2 \right)^{1/2},$$

which squares to yield the result. $\square$

3. Given $z, \zeta \in \mathbb{C}$ such that $|z| < 1$ and $|\zeta| = 1$, define the Poisson kernel (of the disk) as

$$P(z, \zeta) = \frac{1 - |z|^2}{2\pi |z - \zeta|^2}.$$

Prove that

$$\int_0^{2\pi} P(z, e^{it})\, dt = 1.$$

(a) Take the series

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta - t)}$$

and rewrite it as two infinite series—one from $n = 0$ to $n = \infty$ and one from $n = -1$ to $n = -\infty$.

(b) Sum the two (now geometric) series you found to show that

$$P(re^{i\theta}, e^{it}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta - t)}.$$

(c) Swap sum and integral (assume this is valid—it is, but we don't have the background to prove it) to finish the proof.

Proof. First note that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)} &= \sum_{n=-\infty}^{-1} r^{|n|} e^{in(\theta-t)} + \sum_{n=0}^{\infty} r^{|n|} e^{in(\theta-t)} \\
&= \sum_{n=1}^{\infty} \left(re^{i(t-\theta)}\right)^n + \sum_{n=0}^{\infty} \left(re^{i(\theta-t)}\right)^n \\
&= \frac{re^{i(t-\theta)}}{1 - re^{i(t-\theta)}} + \frac{1}{1 - re^{i(\theta-t)}} \\
&= \frac{1 - r^2}{(1 - re^{i(t-\theta)})(1 - re^{i(\theta-t)})} \\
&= \frac{1 - r^2}{|1 - re^{i(t-\theta)}|^2} \\
&= \frac{1 - r^2}{|e^{-i\theta}|^2 \cdot |e^{i\theta} - re^{it}|^2}
\end{aligned}$$

Since $|e^{-i\theta}| = 1$ and $r = |re^{it}|$ we have

$$P(re^{i\theta}, e^{it}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)}.$$

Notice that if $n \neq 0$,

$$\int_0^{2\pi} e^{in(\theta-t)}\, dt = \left.\frac{e^{in(\theta-t)}}{-in}\right|_0^{2\pi} = 0,$$

though if $n = 0$ the integral evaluates to 2π. Finally we have

$$\int_0^{2\pi} P(re^{i\theta}, e^{it})\, dt = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} \int_0^{2\pi} e^{in(\theta-t)} = 1. \qquad \square$$