

QUASI-ISOMETRIES OF HYPERBOLIC SPACE ARE ALMOST ISOMETRIES

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ABSTRACT. In this paper we show that for $n \geq 3$ a quasi-isometry of hyperbolic n -space \mathbb{H}^n to itself is almost an isometry, in the sense that the image of most points on a sphere of radius r are close to a sphere of the same radius. To be more precise, the result is that given $K > 1$ and $\epsilon > 0$ there is a $\delta > 0$ such that the image of any sphere of any radius r under any K -quasi-isometry lies within a distance of δ of another sphere of radius r , except for the image of a proportion ϵ of the source sphere. We illustrate our result with a quasi-isometry of \mathbb{H}^n for which the image of a sphere is the analog of an ellipsoid in Euclidean space. There is no corresponding result when $n = 2$. This failure is illustrated by lifting to the universal cover a surface diffeomorphism which is not isotopic to an isometry.

INTRODUCTION

An example of how quasi-isometries arise in topology is obtained by lifting a homotopy equivalence between compact Riemannian manifolds to the universal covers. This is the starting point for Mostow's proof of the rigidity theorem for hyperbolic manifolds [Mos2]. This theorem asserts the uniqueness, up to isometry, of a hyperbolic structure on a closed manifold of dimension $n \geq 3$. In dimension 2, these hyperbolic structures are not unique but are parametrised by a finite-dimensional Teichmüller space [Ab]. The reason for the difference between dimension 2 and higher dimensions is that quasi-conformal maps of S^m are absolutely continuous with respect to Lebesgue measure, provided $m \geq 2$ [Mos1], but quasi-symmetric maps of S^1 need not be absolutely continuous. In this paper we point out another consequence of (uniform-) absolute continuity; namely that a quasi-isometry of \mathbb{H}^n , $n \geq 3$, almost preserves distance in a very strong sense (1.4). The corresponding statement for \mathbb{H}^2 is false, for example a pseudo-Anosov diffeomorphism of a hyperbolic surface lifts to a quasi-isometry of \mathbb{H}^2 which increases the distance d between typical points by an average amount which is unbounded as d increases (2.2). We give an outline of the proof of the main theorem: Given a quasi-isometry, there is a quasi-conformal

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extension to the sphere at infinity. A quasi-conformal map is differentiable a.e., and is therefore closely approximated by a linear map near a.e. point at infinity. At a.e. point the derivative is non-singular because quasi-conformal maps are absolutely continuous. The behaviour at infinity determines the image of a point in \mathbb{H}^n to within an error which depends only on the quality of the quasi-isometry. This is because a point is determined by the intersection of two geodesics, the images of which are quasi-geodesics, and so are determined to within a bounded error by the behaviour at infinity. A hyperbolic translation can be chosen to have almost the same linear approximation near a given point at infinity. Thus the quasi-isometry is approximated by a translation, near the given point. To obtain a bound on this translation distance (at most points) depending only on K and ϵ but not the particular K -quasi-isometry we use the fact due to Reimann [R] that a K -quasi-conformal map is $\alpha(K)$ -Hölder continuous on the level of measure theory (1.3). This completes the outline.

The proof given here makes vital use of the conformal structure on the sphere at infinity, but perhaps there is a different proof using only hyperbolic geometry. The conformal structure at infinity is not as useful for non-constant negative curvature spaces, but it seems likely that a different proof in the constant curvature case might yield a version of (1.4) for quasi-isometries of negatively curved spaces.

1. THE MAIN RESULT

Definition. A K -quasi-isometry ($K \geq 1$) is a map between metric spaces $\phi : (X, d_X) \rightarrow (Y, d_Y)$ having the following properties:

- (1) $\forall x, x' \in X \quad d_X(x, x') \geq K \Rightarrow K^{-1} \leq \frac{d_Y(\phi x, \phi x')}{d_X(x, x')} \leq K$.
- (2) $\forall y \in Y \quad \exists x \in X \quad d_Y(\phi x, y) \leq K - 1$.

When convenient we shall suppress the constant K which determines the *quality* of the quasi-isometry. Notice that a 1-quasi-isometry is just an isometry. More general definitions can be made (see, for example, [Th]). The notation $S_r(x)$ is used for the sphere of radius r in \mathbb{H}^n centered on x . If Ω is a subset of a metric space X , $N_\delta(\Omega)$ is the neighborhood of Ω consisting of all points in X lying within a distance δ of Ω . A **quasi-geodesic** is the image of the real line under a map which is a quasi-isometry onto its image. Quasi-geodesics have the property that they closely track geodesics:

(1.1) **Proposition** ([Can]). *Given $K \geq 1$ there is a constant $C_1 > 0$ depending only on K such that if $\phi : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is a K -quasi-isometry, then for every geodesic γ there is another geodesic γ' such that $\phi(\gamma)$ lies within a distance C_1 of γ' .*

It is well known [Mos1] that a K -quasi-isometry $\phi : \mathbb{H}^n \rightarrow \mathbb{H}^n$ has a K^2 -quasi-conformal extension $\bar{\phi}$ to the sphere at infinity S_∞^{n-1} (quasi-symmetric if $n = 2$). This has the consequence that the image under a K -quasi-isometry ϕ of a point $x \in \mathbb{H}^n$ is determined to within a distance C_2 (depending only on K) by the action of $\bar{\phi}$ on S_∞^{n-1} . Indeed, choose two orthogonal geodesics γ_1 and γ_2 through x ; then there are uniquely determined geodesics γ'_1 and γ'_2 such that $\phi(x)$ lies in the intersection of the neighborhoods of size C_1 around γ'_1 and γ'_2 (see Figure 1). The diameter of this intersection is bounded above in terms of C_1 and of the length of the subarc of γ'_1 which lies within a distance

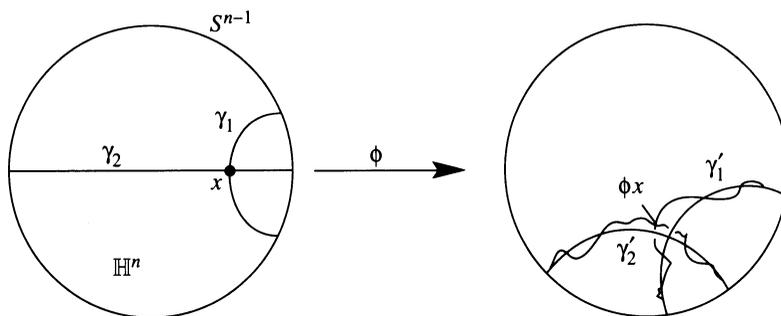


FIGURE 1

$2C_1$ of γ'_2 . The length of this subarc is bounded in terms of K because if γ'_1 and γ'_2 are close along a long subarc, then there must be corresponding long arcs on γ_1 and γ_2 which are close to each other. But this would contradict the fact that γ_1 and γ_2 are orthogonal. The geodesics γ'_1 and γ'_2 are determined by the images under $\bar{\phi}$ of the endpoints of γ_1 and γ_2 . However it suffices to know the images of only three of these endpoints in order to know the image of x to within a bounded error. This is because the argument above can be modified to use a geodesic ray orthogonal to x in place of γ_2 . This establishes the following lemma.

(1.2) **Lemma.** *Given $K \geq 1$ there is $C_3 \geq 0$ such that if ϕ and τ are any two K -quasi-isometries of \mathbb{H}^n which agree on three of the endpoints of two orthogonal geodesics passing through a given point x in \mathbb{H}^n , then the distance between $\phi(x)$ and $\tau(x)$ is bounded by C_3 .*

A map $\phi : M \rightarrow N$ between Riemannian manifolds is said to be **closely approximated by its derivative** in a δ -neighborhood of $x \in M$ if the following condition holds:

$$\forall u \in T_x M \quad \|u\| \leq \delta \implies \|\exp_{\phi(x)}^{-1}(\phi \exp_x u) - \phi'(x)u\| \leq \|\phi'(x)u\|/1000.$$

Definition. We will use $\lambda(\Omega)$ for the m -dimensional Lebesgue measure of the subset $\Omega \subset S^m$ of any m -sphere **normalised** so that $\lambda(S^m) = 1$.

Given $\epsilon > 0$ we claim there is a subset Ω of S_∞^{n-1} and constants $L, \ell, \delta > 0$ with the following properties:

- (1) $\lambda(S_\infty^{n-1} - \Omega) < \epsilon$.
- (2) $\bar{\phi}$ is differentiable at each $x \in \Omega$.
- (3) Every directional derivative of $\bar{\phi}$ is bounded in norm below by ℓ and above by L at every point of Ω .
- (4) $\bar{\phi}$ is closely approximated by its derivative in a δ -neighborhood of each $x \in \Omega$.

To see this, since $\bar{\phi}$ is quasi-conformal, it is differentiable a.e. [Ahl], giving condition (2). The directional derivatives are measurable functions, and therefore L exists. Also $\bar{\phi}$ is quasi-conformal, and therefore absolutely continuous with respect to Lebesgue measure [Ahl] so that the Jacobian of $\bar{\phi}$ is non-zero a.e. implying $\ell > 0$ exists and proving (3). (4) is proved by the following standard

argument in measure theory. Given a point $x \in S_\infty^{n-1}$ at which $\bar{\phi}$ is differentiable, there is $\delta(x) > 0$ such that $\bar{\phi}$ is closely approximated by its derivative in a $\delta(x)$ -neighborhood. For each integer $p > 0$ set

$$V_p = \{x \in S_\infty^{n-1} : \bar{\phi} \text{ is closely approximated by its derivative in a } (1/p)\text{-neighborhood of } x\}.$$

Assuming for the moment that V_p is measurable, since V_p is an increasing sequence converging to a set of full measure, it follows that for p sufficiently large, $\lambda(S_\infty^{n-1} - V_p) < \epsilon$ thus $\Omega = \bigcap V_p$ satisfies (4). To show V_p is a measurable set choose a countable dense subset $\{a_i\}$ of the ball of radius $1/p$ in \mathbb{R}^{n-1} . Pick a point $y \in S_\infty^{n-1}$ and choose a trivialisation of $T_*(S_\infty^{n-1} - y)$. Use this trivialisation to continuously identify a_i with a point $a_i(x)$ in the ball of radius $1/p$ in $T_x S_\infty^{n-1}$. Since $\phi'(x)$ is measurable, there is a measurable function $f_i : (S_\infty^{n-1} - y) \rightarrow \mathbb{R}$ defined by

$$f_i(x) = \|\exp_{\phi(x)}^{-1}(\phi \exp_x a_i(x)) - \phi'(x)a_i(x)\|/\|\phi'(x)a_i(x)\|.$$

Thus $V_{p,i} = \{x \in S_\infty^{n-1} : f_i(x) \leq 1/1000\}$ is measurable, and it follows that $V_p = \bigcap_{i=1}^\infty V_{p,i} \pm \{y\}$ is measurable. This establishes that Ω exists with the above properties.

Fix a point $x \in \mathbb{H}^n$ and compose ϕ with an isometry so that $\phi(x) = x$. Choose coordinates in the Poincaré disc model so that x is at the origin. Let Ω be the subset of S_∞^{n-1} with properties (1)–(4) (relative to Lebesgue measure λ on S_∞^{n-1} given by the chosen coordinatisation). Given any point $a \in \Omega$, let γ_1 be the geodesic in \mathbb{H}^n through x ending on a . By replacing ϕ by the composition of ϕ with a suitable rotation centred on the origin we may suppose that $\bar{\phi}(a) = a$. Let y be the point on γ_1 a distance r from x towards a (see Figure 2). Choose a geodesic γ_2 passing through y and orthogonal to γ_1 , and let b_1 and b_2 be the endpoints of γ_2 on S_∞^{n-1} . For r sufficiently large depending on δ , $\bar{\phi}$ is closely approximated by its derivative at a in a neighborhood which includes b_1 and b_2 . Let τ be a hyperbolic translation with axis γ_1 which maps b_1 to $\bar{\phi}(b_1)$.

We would like to apply Lemma (1.2) to the maps ϕ and τ using the three points a, b_1, b_2 ; now $\phi(a) = \tau(a) = a$ and $\phi(b_1) = \tau(b_1)$ but $\phi(b_2) \neq \tau(b_2)$.

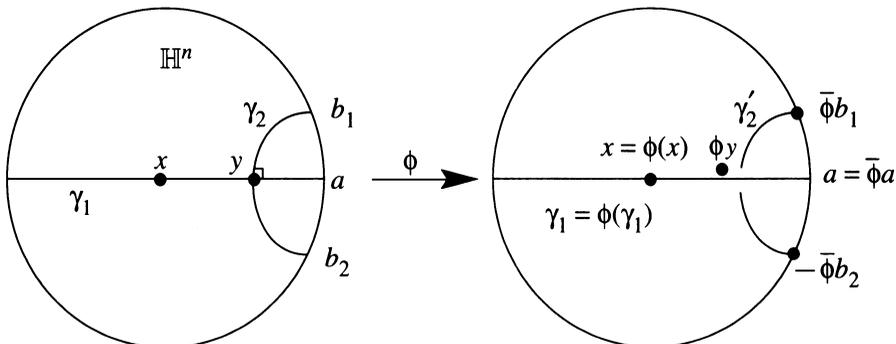


FIGURE 2

However ϕ is closely approximated by its derivative in a neighborhood of a which includes b_2 , therefore $\phi(b_2)$ and $\tau(b_2)$ are very close in the appropriate sense. To be precise there is an isometry τ' very close to the identity which fixes both a and $\phi(b_1)$ and such that $\tau'(\tau b_2) = \phi(b_2)$ and τ' moves τy a small distance. The isometry τ' is a loxodromic with axis whose endpoints are a and ϕb_1 . We replace τ by $\tau' \circ \tau$ and apply lemma (1.2) to ϕ and τ with the points a, b_1, b_2 .

Then by Lemma (1.2) the hyperbolic distance between $\tau(y)$ and $\phi(y)$ is bounded above by C_3 . The translation distance of τ is determined to within a bounded error by any directional derivative of $\bar{\phi}$ at a , and is thus bounded above in terms of ℓ and L . Thus the distance which τ moves y is bounded by a constant $C(K, \ell, L)$ depending on $K, \ell(\phi, \epsilon, x)$ and $L(\phi, \epsilon, x)$. Using (1.2) this implies that $|d(\phi x, \phi y) - d(x, y)| \leq C(K, \phi, \epsilon, x) + C_3$. This estimate applies to the point $y \in S_r(x)$ which lies on any ray from x to any point $a \in \Omega$. Such points form a subset of $S_r(x)$ whose complement has measure at most ϵ . Finally we show that $C(K, \phi, \epsilon, x)$ can be bounded independently of ϕ and x . If this is not the case then there is a sequence of K -quasi-isometries ϕ_i for which the smallest such constant goes to infinity. This implies that there is a sequence $r_i \rightarrow \infty$ and $\Gamma_i \subset S_{r_i}(x)$ with $\lambda(\Gamma_i) \geq \epsilon$ and $d(\phi_i(\Gamma_i), x) - r_i \rightarrow \infty$. Use radial projection from x onto S_∞^{n-1} to identify Γ_i with a subset $\bar{\Gamma}_i \subset S_\infty^{n-1}$. Then $\lambda(\bar{\Gamma}_i) \geq \epsilon$ but the image under the K^2 -quasi-conformal map $\bar{\phi}_i$ has $\lambda(\bar{\phi}_i \bar{\Gamma}_i) \rightarrow 0$ as $i \rightarrow \infty$. The following result is an immediate consequence of [R], Corollary, page 262. It says that on the level of measure theory K -quasi-conformal maps which fix three points are uniformly Hölder continuous.

(1.3) **Theorem [R].** *Given $K > 1, m \geq 2$ and three points on S^m there are constants $\alpha > 0$ and $\beta > 0$ such that for all K -quasi-conformal maps $\phi : S^m \rightarrow S^m$ which fix the three specified points the following holds:*

$$\text{for all measurable } \Gamma \subset S^m \quad \lambda(\phi\Gamma) \leq \beta \cdot (\lambda(\Gamma))^\alpha.$$

We wish to apply this with $\phi = \bar{\phi}_i^{-1}$ and $\Gamma = \bar{\phi}_i \bar{\Gamma}_i$. The inverse of a K -quasi-conformal map is also K -quasi-conformal, and the requirement that three points be fixed can be met as follows. The sequence of K -quasi-isometries ϕ_i all fix x and therefore has a subsequence which converges on compacta. This implies the images of the three points under $\bar{\phi}_i$ converge to distinct points. Choose a sequence of conformal maps τ_i of S_∞^{n-1} which converge uniformly and which agree with $\bar{\phi}_i$ on the three points. Then $\tau_i^{-1} \circ \bar{\phi}_i$ fixes the three points. Replace $\bar{\phi}_i$ with this map and apply (1.3) to get a contradiction. Thus there is a bound on the translation distance which can be chosen depending only on K, n and ϵ . We have now established:

(1.4) **Main Theorem.** *Given $K > 1, \epsilon > 0$ and $n \geq 3$ there is a constant $\delta > 0$ such that for every K -quasi-isometry $\phi : \mathbb{H}^n \rightarrow \mathbb{H}^n$ and any $r > 0$ there is a subset $\Omega \subset S_r(x)$ with $\lambda(S_r(x) - \Omega) \leq \epsilon \lambda(S_r(x))$ and $\phi(\Omega) \subset N_\delta(S_r(\phi x))$.*

An example. Choose a geodesic γ in \mathbb{H}^n and a point x on γ , and define for $K > 1$ a K -quasi-isometry ϕ as follows. Isometrically identify γ with \mathbb{R} so that x is identified with 0; then ϕ is to map γ to itself by multiplication by K . Extend this over the rest of \mathbb{H}^n by foliating \mathbb{H}^n by codimension-1 hyperplanes

orthogonal to γ and having ϕ preserve this foliation, mapping one hyperplane to another by means of parallel translation along γ . The corresponding map of Euclidean space sends a sphere to an ellipsoid of eccentricity K , so that the image of a sphere in \mathbb{H}^n centered on x is analogous in this sense to an ellipsoid. To see that ϕ is K -Lipschitz, observe that the tangent space to \mathbb{H}^n at each point splits orthogonally as $T_*\mathbb{H}^n \cong T_*\mathbb{H}^{n-1} \oplus V$ where V is a 1-dimensional space and this splitting is ϕ -invariant. Then $D\phi|T_*\mathbb{H}^{n-1}$ is the identity, and orthogonal projection onto γ shows that $D\phi|V$ is multiplication by K . In the Poincaré disc model, one may picture $\phi(S_r(x))$ as being very close (in the hyperbolic metric) to $S_r(x)$ everywhere except in a small (in the Euclidean metric) neighborhood of γ , where it has long (in the hyperbolic metric) spikes jutting outwards. For r large, the pre-image of the spikes in the unit tangent sphere at x has small $(n-1)$ -measure.

2. THE CASE OF DIMENSION 2

The purpose of this section is to show that there is no result corresponding to Theorem (1.4) for quasi-isometries of the hyperbolic plane. Choose a homeomorphism ϕ of a closed hyperbolic surface F and a lift, $\tilde{\phi}$, of this to the universal cover \mathbb{H}^2 . The following is a consequence of (22.14) in Mostow's book [Mos2], page 178, which is also proved in [Ag].

(2.1) **Proposition.** *The extension of $\tilde{\phi}$ to the circle at infinity $\bar{\phi}: S^1_\infty \rightarrow S^1_\infty$ has finite non-zero derivative at some point if and only if ϕ is isotopic to an isometry of F which is necessarily of finite order.*

We will now assume that ϕ is not isotopic to an isometry (e.g. ϕ is pseudo-Anosov). Fix a point $x \in \mathbb{H}^2$. Then by composing $\tilde{\phi}$ with a suitable isometry, we may suppose that $\tilde{\phi}(x) = x$. Since $\bar{\phi}$ is a homeomorphism of S^1_∞ , it is differentiable a.e., which by (2.1) implies that $\bar{\phi}$ has zero derivative a.e. It remains to show that this is impossible for a quasi-isometry satisfying the conclusion of Theorem (1.4). Let $\Omega \subset S^1_\infty$ be the set of points at which $\bar{\phi}$ is differentiable. Given $a \in \Omega$ there is a connected neighborhood U_a of a in which $\bar{\phi}$ is ϵ -closely approximated by its derivative, a constant function, in the following sense:

$$\forall b \in U_a \quad b \neq a \rightarrow \|\exp_{\bar{\phi}(a)}^{-1}(\bar{\phi}b)\|/\|\exp_a^{-1}(b)\| \leq \epsilon.$$

Consider a geodesic γ with endpoints in U_a on opposite sides of a and let y be the point on γ closest to x . It follows from (1.2) that $d(\phi x, \phi y) > d(x, y) + C_\epsilon$ where $C_\epsilon \rightarrow \infty$ as $\epsilon \rightarrow 0$. Denote by \mathcal{U}_a the (Euclidean-) measurable subset of $\mathbb{H}^2 \cup S^1_\infty$ consisting of U_a and all such points y . Let $V = \bigcup_{a \in \Omega} \mathcal{U}_a$. Then $\tilde{\phi}(V \cap S_r(x))$ lies entirely outside $S_{r+C_\epsilon}(x)$. Since V is measurable, $\lambda(V \cap S_r(x)) \rightarrow \lambda(\Omega) = 1$ as $r \rightarrow \infty$. But by making ϵ arbitrarily small we can make C_ϵ arbitrarily large and this violates the conclusion of (1.4). We have now established:

(2.2) **Theorem.** *Suppose ϕ is a diffeomorphism of a closed hyperbolic surface which is not isotopic to an isometry and that $\tilde{\phi}$ is any lift of ϕ to \mathbb{H}^2 . Then given $x \in \mathbb{H}^2$, $\delta > 0$ and $\epsilon > 0$ for all sufficiently large r there is a subset $\Omega \subset S_r(x)$ with $\lambda(S_r(x) - \Omega) \leq \epsilon \lambda(S_r(x))$ and $\tilde{\phi}(\Omega) \cap N_\delta(S_r(\tilde{\phi}x)) = \emptyset$.*

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