Mathematical Research Letters **3**, 51–60 (1996)

COMBINATORIAL SCALAR CURVATURE AND RIGIDITY OF BALL PACKINGS

Daryl Cooper and Igor Rivin

Introduction

Let M^3 be a triangulated three-dimensional manifold. In this paper we define a combinatorial analogue of scalar curvature for M^3, and also a combinatorial analogue of conformal deformation of the metric. We further define a functional $\mathcal{S}$ on the combinatorial conformal deformation space, show that $\mathcal{S}$ is concave, and show that critical points of $\mathcal{S}$ correspond precisely to metrics of constant combinatorial scalar curvature on M^3. These results are then applied to showing rigidity of ball packings with prescribed combinatorics (the concepts are quite similar to Colin de Verdière's work on circle packing of surfaces [2]. See also [5] for a related variational argument).

The plan of the paper is as follows. In section 1 we define the class of *conformal simplices* in $\mathbb{E}^3$, and prove the necessary local versions of our results. In section 3 we extend these techniques to conformal simplices in $\mathbb{H}^3$. In section 4 we study the deformation space of conformal simplices. In section 5 we prove the results on scalar curvature alluded to above, and in section 6 we discuss the ball-packing results.

1. Conformal simplices

Let T be a simplex in $\mathbb{E}^3$. Denote the vertices of T by $v_1, \dots, v_4$, and denote the edge joining v_i and v_j by e_{ij}. The length of e_{ij} shall be denoted by l_{ij}, while the dihedral angle of e_{ij} shall be denoted by α_{ij}. The solid angle at the vertex v_i of T shall be denoted by S_i. From the Gauss-Bonnet formula we know that $S_i = \sum_{j \neq i} \alpha_{ij} - \pi$.

We say that T is *conformal* if there are $r_1, \dots, r_4 > 0$ such that $l(e_{ij}) = r_i + r_j$. Denote the space of all conformal simplices by $\mathcal{C}_T$ – this is a topological space, if we require that the map of $\mathcal{C}$ to $\mathbb{R}^4$, where we map a conformal sequence to the radii $r_1, \dots, r_4$ defining it, be a homeomorphism. The collection of vectors $(r_1, \dots, r_4) \in \mathbb{R}^4$ which correspond to geometric simplices

Received October 26, 1995.

is an open set. Denote this set by $\mathcal{C}_T^{\mathbf{r}}$. Since the set of all three-dimensional Euclidean simplices is an open set in $\mathbb{R}^6$ (as parametrized by edge-lengths, for example), it is clear that the property of conformality is rather special.

In the sequel the simplex T and all of its geometric parameters will be viewed as functions of $\mathbf{r} = (r_1, r_2, r_3, r_4) \in \mathcal{C}_T^{\mathbf{r}}$. Let

$$S(\mathbf{r}) = \sum_{i=1}^{4} S_i r_i.$$

The next two results form the foundation of what follows.

Lemma 1.1.

$$dS = \sum_{i=1}^{4} S_i dr_i.$$

Proof. Let us rewrite the formula for S using the relations $l_{ij} = r_i + r_j$ and $S_i = \sum_{j \neq i} \alpha_{ij} - \pi$. A short computation shows that

$$S = \sum_{i \neq j} l_{ij}\alpha_{ij} - \pi \sum_{i=1}^{4} r_i,$$

and thus

$$dS = \sum_{i \neq j} \alpha_{ij} dl_{ij} + \sum_{i \neq j} l_{ij} d\alpha_{ij} - \pi \sum_{i=1}^{4} dr_i.$$

However, by the Euclidean case of the Schläfli differential formula, the second sum above vanishes, and since $l_{ij} = r_i + r_j$, the claim of the lemma follows. □

Note. The Schläfli differential formula (for a good exposition and proof, see [1][Chapter 7, section 2.2.2]) states that as a simplex T (and thus any polyhedron) varies smoothly in one of the spaces of constant curvature $\mathbb{H}^n$, $\mathbb{S}^n$ or $\mathbb{E}^n$, its volume also varies smoothly, and furthermore

$$K d \operatorname{vol} T = \frac{1}{n-1} \sum_{\dim F = n-2} \operatorname{vol} F \, d\alpha_F, \tag{1.1}$$

where K is the curvature of the space, and α_F is the dihedral angle at the $(n-2)$-dimensional face F. Note that when $K = 0$, the formula simply says that the right-hand side vanishes.

Lemma 1.2. *The Hessian H of S is negative semi-definite – its null-space is spanned by the vector* $\mathbf{r}$.

Proof. It is clear that $\mathbf{r}$ is in the null-space of the Hessian, since it corresponds to a rescaling of the simplex. To show that all other eigenvalues of H are negative it is enough to show that the diagonal entries of H are negative, while the off-diagonal elements are positive. To see this, let $(v_1, \ldots, v_4)$ be an eigenvector of H corresponding to a non-negative eigenvalue λ, and let $w_i = v_i/r_i$. We want to show that $w_i = w_j$, for all i, j. If not, assume without loss of generality that $0 < w_1 = \max\{w_i\}$. Then, $(\lambda - H_{11})v_1 = \sum_{i>1} H_{1i}v_i$, so

$$\lambda r_1 w_1 - H_{11} r_1 w_1 = \sum_{i>1} H_{1i} r_i w_i \leq \sum_{i>1} H_{1i} r_i w_1,$$

and so $\lambda \leq 0$, with equality if and only if all of the w_i are equal.

Note now that $H_{ij} = \partial S_i / \partial r_j$. If r_1, r_2, r_3 are kept fixed and r_4 is increased, v_4 is moving along a trajectory h such that the angles between the tangent vector to h and the edges e_{4i} are equal (an easy consequence of the law of cosines). Thus, the simplex $T(r_1, r_2, r_3, r_4 + \epsilon)$ (where $\epsilon > 0$) strictly contains $T(r_1, r_2, r_3, r_4)$, while sharing its base. The desired inequalities are then clear. □

As a preview of the argument to come in the next section, we prove the following results.

Theorem 1.3. *Fix the scale of the simplices under consideration, by setting*

$$r_1 + r_2 + r_3 + r_4 = 1. \tag{1.2}$$

Then the regular simplex, for which $r_i = 1/4$, is a critical point of the function S, and furthermore, the regular simplex cannot be conformally deformed while keeping the solid angles fixed.

Proof. By Lemma 1.2, the function S is strictly concave on the slice of $\mathcal{C}_T^{\mathbf{r}}$ determined by the equation 1.2. By the principle of Lagrange multipliers and Lemma 1.1, S has a critical point subject to equation 1.2, precisely when all of the solid angles are equal. By concavity, such a critical point is isolated, proving the rigidity statement. □

A similar argument shows the following:

Theorem 1.4. *A conformal simplex cannot be deformed while keeping the solid angles fixed; that is, the set $\mathcal{C}_{S_1,S_2,S_3,S_4} \subset \mathcal{C}_1$ is discrete.*

Proof. We have seen above that this is true for the regular simplex. To show the theorem, we modify the function S as follows: Let $S_{a_1,a_2,a_3,a_4} = S + \sum a_i r_i$. Now S_{a_1,a_2,a_3,a_4} is still strictly concave subject to the condition 1.2, but its gradient is $(S_1 + a_1, \ldots, S_4 + a_4)$. Hence the critical points of $S_{a_1,\ldots,a_4}$ occurs when $S_i + a_i = S_j + a_j$, for all $i, j \leq 4$. This shows the assertion of the theorem. □

2. Another coordinate system

While the radii r_i are a natural set of coordinates for the space of conformal simplices, it turns out that, in many ways, the *curvatures* of the spheres are better. That is, instead of $r_1, \ldots, r_4$, use $\kappa_1, \ldots, \kappa_4$, where $\kappa_i r_i = 1$. Using this parametrization embeds $\mathcal{C}_T$ as an open set in $\mathbb{R}^4$. Call this open set $\mathcal{C}_T^\kappa$.

Consider again the function S, as in Section 1. As a function of $\kappa = (\kappa_1, \ldots, \kappa_4)$, S can be written as

$$S = \sum_{i=1}^{4} S_i/\kappa_i. \tag{2.1}$$

The following lemma is shown the same way as in Lemma 1.1.

Lemma 2.1.

$$dS = -\sum_{i=1}^{4} \frac{S_i}{\kappa_i^2} d\kappa_i.$$

The following result follows from the proof of Lemma 1.2

Lemma 2.2. *The Hessian H of S with respect to κ is negative semi-definite – its null-space is spanned by the vector κ.*

Proof. As before, it is clear that κ is in the null-space of H, since this corresponds to rescaling the simplex. Now, differentiating Lemma 2.1, it is immediate that

$$H_{ij} = \frac{\partial S_i}{\partial \kappa_j} = \frac{\partial S_i}{\partial r_j} r_i^2 r_j^2,$$

when $i \neq j$. In particular, H_{ij} is positive for $i \neq j$, by the proof of Lemma 1.2, and the algebraic part of that proof now goes through unchanged to complete the proof of Lemma 2.2. □

3. Hyperbolic conformal simplices

In this section we consider conformal simplices (defined as in Section 1) in hyperbolic 3-space $\mathbb{H}^3$, rather than in $\mathbb{E}^3$. Rather than repeating all of the definitions, we will simply point out the differences. The main one of these is in the definition of the function S. We will now define S_h as

$$S_h(\mathbf{r}) = \sum S_i r_i + 2 \ \operatorname{vol} T. \tag{3.1}$$

Then, Schläfli's formula 1.1 immediately implies the following analogue of Lemma 1.1:

Lemma 3.1.

$$dS_h = \sum_{i=1}^{4} S_i dr_i.$$

The following is the hyperbolic analogue of Lemma 1.2:

Lemma 3.2. *The Hessian H_h of S_h is negative definite.*

Proof. The proof of the lemma is similar to the proof of Lemma 1.2. The first observation is that as r_1, r_2, r_3 are kept fixed and r_4 is increasing, v_4 is moving along a trajectory h such that the angles between the tangent vector to h and the edges e_{4i} are equal (an easy computation using the hyperbolic law of cosines). Thus, the simplex $T(r_1, r_2, r_3, r_4 + \epsilon)$ ($\epsilon > 0$) strictly contains $T(r_1, r_2, r_3, r_4)$, while sharing its base. We see that $S_i(r_1, r_2, r_3, r_4 + \epsilon) > S_i(r_1, r_2, r_3, r_4)$, for $i < 4$, so

$$\partial S_i / \partial r_j > 0, \qquad i \neq j. \tag{3.2}$$

By containment, vol $T(r_1, r_2, r_3, r_4 + \epsilon) >$ vol $T(r_1, r_2, r_3, r_4)$, and so, by the Schläfli formula 1.1,

$$\sum_{j=1}^{4} r_j \frac{\partial S_j}{\partial r_i} < 0, \qquad i = 1, \ldots, 4. \tag{3.3}$$

Equations 3.2 and 3.3 imply that H_h is negative definite by Gershgorin's theorem (or the reader is welcome to modify the argument in the proof of Lemma 1.2). □

4. The space of conformal simplices

In this section we will describe the space of Euclidean conformal simplices $\mathcal{C}_T$ introduced in section 1. The first observation is the following:

Lemma 4.1. *A simplex T is conformal if and only if there exists a unique sphere s_T which is tangent to all of the edges of T. Furthermore, the point of tangency of s_T with the edge e_{ij} is at distance r_i from v_i.*

Proof. We will show this result in all dimensions greater than 1. The "if" direction is trivial. We will show the "only if" direction by induction on dimension. In dimension 2, where all triangles are conformal, the result is obvious.

Now let T be a conformal simplex with radii r_i. First note that there is a *unique* sphere s tangent to e_{1j}, $j \neq 1$. such that the points of tangency are r_1 away from v_1. Consider the intersection of s with one of the faces of T containing v_1 – by induction we know that that the lower dimensional sphere is tangent to all of the edges of that face. Since all edges of T belong to a face containing v_1, the lemma follows. $\square$

Now, first normalize so that the radius of s_t is equal to 1, and consider the pattern of circles formed on s_T by the spheres of radii r_i centered at the vertices of T. These circles are mutually tangent. Conversely, any such pattern of circles C determines a conformal simplex. Since for any two circle packings C_1 and C_2 of the sphere $\mathbb{S}^2$ whose nerve is the simplex graph, there is a conformal transformation sending C_1 to C_2 (this is elementary geometry, not requiring the Andreev-Thurston theorem; in particular, it also works in any dimension), the set of such packings can be naturally identified with the group of conformal transformations of the sphere – $PSL(2, \mathbb{C})$. Since we are only interested in the congruence class of the conformal simplex T, and are no interested in which way it is pointing, we must further normalize by factoring out the rotation group $SO(3)$. What we have just determined is:

Theorem 4.2. *The space of* isometry classes *of Euclidean conformal simplices $\mathcal{C}_T$ can be naturally identified with the hyperbolic space $\mathbb{H}^3$. In particular, $\mathcal{C}_T$ is homeomorphic to the ball $B^4 = \mathbb{H}^3 \times \mathbb{R}$.*

Note. The discussion above applies just as well to hyperbolic conformal simplices, except that since we can no longer normalize by fixing the radius of the mid-sphere s_t, and so $\mathcal{C}_H \simeq \mathbb{H}^3 \times \mathbb{R}^+$, where $\mathcal{C}_H$ is the deformation space of hyperbolic conformal simplices.

It will be useful to know exactly how $\mathcal{C}_T$ is parametrized by our coordinates, in particular, what the boundary of $\mathcal{C}_T$ (as a subset of $\mathbb{R}^4$) looks like. Note that while for any three positive real numbers r_1, r_2, and r_3, the sums $r_1 + r_2$, $r_1 + r_3$, and $r_2 + r_3$ satisfy the triangle inequality, not every positive quadruple $\mathbf{r} = (r_1, r_2, r_3, r_4)$ defines a geometric conformal simplex. Degeneracy occurs when r_4 is large enough so that the sphere of radius r_4 is large enough to be tangent to the other three spheres, yet small enough that its center v_4 lies in the plane defined by v_1, v_2, and v_3. This condition is defined by Soddy's theorem – below we state a version for arbitrary dimension, recently obtained[1] by M. Kovalev [3].

Theorem 4.3. *Let S_i, $i = 1, \ldots, n+2$, be $n+2$ pairwise tangent distinct spheres in the n-dimensional Euclidean space E^n, such that S_i has radius R_i. Then*

$$n \sum_{i=1}^{n+2} K_i^2 = \left(\sum_{i=1}^{n+2} K_i \right)^2, \tag{4.1}$$

where $K_i = 1/R_i$, if S_i does not contain the other spheres, otherwise $K_i = -1/R_i$.

While the above theorem is generally quite useful, it is not necessary to obtain the following observation:

Theorem 4.4. *The space $\mathcal{C}_T \subset \mathbb{R}^4$ is not convex.*

Proof. Consider the section $\mathcal{C}_T^{1,1}$ of $\mathcal{C}_T$ defined by $r_1 = r_2 = 1$. We can think of $\mathcal{C}_T^{1,1}$ as a subset of a plane. As described above, one component ∂_1 of the boundary of $\mathcal{C}_T^{1,1}$ is defined by the condition that there are four mutually-tangent circles in the plane, of radii $1, 1, r_3, r_4$ – $\mathcal{C}_T^{1,1}$ itself lies above ∂_1. Notice that as r_4 approaches infinity, r_3 also grows, but to a finite limit, and so ∂_1 is not convex. □

Note. Using Theorem 4.3, it can actually be shown that ∂_1, as defined in the proof above is strictly *concave.*

5. Conformal deformation space of singular Euclidean metrics

Consider a closed three-dimensional manifold M^3, together with a topological triangulation $\mathcal{T}$. If we equip all of the simplices of $\mathcal{T}$ with metrics, so that they are isometric to geodesic simplices in $\mathbb{E}^3$, and, furthermore, these metrics are compatible (which is to say that all of the gluing maps are Euclidean isometries), M^3 acquires a metric, which is flat on the complement of the 1-skeleton of $\mathcal{T}$. We define the *solid angle* S_v of $(M^3, \mathcal{T})$ at a vertex v to be the sum of the solid angles of the simplices of $\mathcal{T}$ incident to v at the vertex identified to v. The quantity $4\pi - S_v$ is a combinatorial analogue of scalar curvature, in that it measures the difference in, on the

[1]According to the translator's note in the English translation of Kovalev's paper, Theorem 4.3 was already known to Coxeter in the thirties.

one hand, the growth rate of a small ball centered at v in M^3 and, on the other hand, the growth rate of the volume of an Euclidean ball of the same radius.

Just as in the previous section we use the notation e_{ij} for the edge joining adjacent vertices v_i and v_j of $\mathcal{T}$, l_{ij} for the length of that edge, and α_{ij} for the total cone angle at that edge.

We say that a singular Euclidean structure on $(M^3, \mathcal{T})$ is *conformal*, if there exists a function $\mathbf{r} : V(\mathcal{T}) \to \mathbb{R}^+$, such that (denoting $\mathbf{r}(v_i)$ by r_i), $l_{ij} = r_i + r_j$. Note that a conformal structure always exists, since we can make all simplices regular and of the same size. The term "conformal" can be justified – we think of a general conformal structure as obtained by rescaling the "metric" of this regular one.

Let $\mathcal{C} \subset \mathbb{R}^{|V(\mathcal{T})|}$ be the set of those $\mathbf{r}$ which correspond to all the simplices of $\mathcal{T}$ being geometric and non-degenerate – it's clear that $\mathcal{C}$ is an open set in $\mathbb{R}^{|V(\mathcal{T})|}$. As before, we will think of geometric invariants of $(M^3, \mathcal{T})$ as functions on $\mathcal{C}$.

Let $\mathcal{S} = \sum_{v \in V(\mathcal{T})} S_v r_v$. It is clear that $\mathcal{S} = \sum_{T \in \mathcal{T}} S_T$, where S_T is the function defined in section 1. It is then immediate that $\mathcal{S}$ is weakly concave on $\mathcal{C}$, and if in addition we restrict our attention to the slice $\mathcal{C}_1$ of $\mathcal{C}$ given by the condition

$$\sum r_i = 1, \tag{5.1}$$

then S_T is strictly concave.

Now, the proof of Theorem 1.4 goes through unchanged to show

Theorem 5.1. *A conformal structure on $(M^3, \mathcal{T})$ cannot be deformed (except by rescaling) while keeping the solid angles at the vertices of $\mathcal{T}$ fixed; that is, the set of conformal structures with prescribed solid angles is a discrete subset of $\mathcal{C}_1$.*

Note. The proof actually shows something a bit stronger than Theorem 5.1 – we need only keep the *differences* between the solid angles fixed.

Remark 5.2. By the results in section 3 everything in this section, with the exception of the preceding Note, translates essentially verbatim to the case of *hyperbolic conformal structures* on M^3 (those where the simplices of $\mathcal{T}$ are hyperbolic confromal simplices).

6. Applications to ball packing

Consider the special case of the situation in the previous section where the metric on $(M, \mathcal{T})$ is in fact non-singular Euclidean or hyperbolic in the interior (though the boundary may be polyhedral). The conformality of the structure means that there are Euclidean (resp. hyperbolic) balls centered at the vertices of $\mathcal{T}$, such that the radius of the ball B_v centered at v is equal to r_v. Two of such balls B_v and B_w are tangent whenever v and w share an edge, and Theorem 5.1 tells us:

Theorem 6.1. *Let M^3 be a Euclidean (hyperbolic) 3-manifold with polyhedral boundary. A ball packing whose nerve is a triangulation $\mathcal{T}$ of M is cannot be deformed (except by rescaling in the Euclidean case) while keeping the solid angles at the boundary of M fixed.*

Note. If M is a closed manifold the clause following "while' is vacuous. The theorem says that the sets of structures corresponding to ball-packings are discrete subsets of $\mathcal{C}_1$ and $\mathcal{C}_H$ respectively.

A particularly simple example of a Euclidean manifold with boundary is a polyhedron in $\mathbb{R}^n$. Since one can always stereographically project the sphere S^n onto R^n in such a way that $n+1$ mutually tangent spheres go to spheres all of radius one (as observed in [2]), the above theorem implies:

Theorem 6.2. *The geometry of ball-packing of the sphere S^3 whose nerve is a triangulation $\mathcal{T}$ is rigid up to Möbius transformations.*

7. Remarks and questions

The results above can be viewed as generalizing of the work of Colin de Verdière [2] to higher dimensions. A rather difficult problem (perhaps intractable in dimension greater than 3) is obtaining any sort of existence results. It is known that no analogue of the results in two dimensions (that any triangulation of a surface is the nerve a circle packing) exists (see [4]), but no characterization is known. Indeed, it is not obvious to the authors that there is an infinite family of ball-packings of S^3. Nor is it obvious that every hyperbolic manifold admits even one ball-packing. A characterization of those Euclidean 3 manifolds (even those homeomorphic to the torus T^3) which admit ball-packings is also somewhat elusive (the results above indicate that only a countable family of metrics on T^3 can be packed by balls).

The hypothesis in sections 5 and 6 requiring M^3 to be a manifold is clearly too strong – almost any semi-simplicial complex works just as well, though there are some degenerate examples, like two simplices joined at the vertices only. A necessary and sufficient condition is that if, for each connected component of M we form a graph $G_{\mathcal{T}}$, whose vertices are top-dimensional faces of $\mathcal{T}$ and such that two vertices are joined by an edge whenever the corresponding simplices share an edge, then $G_{\mathcal{T}}$ must be connected.

The non-convexity of $\mathcal{C}$, as shown in section 4 shows that the current methods do not suffice to give global uniqueness in Theorems 5.1 and 6.1. On the other hand, the functional $\mathcal{S}$ turns out to be convex with respect to the "curvature" coordinates K_i as used in Soddy's theorem 4.3. This analysis shows that K has a *unique maximum* on $\mathcal{C}$, though there may be other critical points, since the analysis has to be performed on a hypersurface in $\mathcal{C}$ [6].

Acknowledgements

The authors would like to thanks Oded Schramm for pointing out an oversight in a previous version of this paper. The second author would like to thank Melbourne University for its hospitality.

References

1. D. V. Alekseevskij, E. Vinberg, and A. S. Solodovnikov, *Geometry of spaces of constant curvature*, Geometry II, EMS **29**; Springer Verlag, Berlin (1993).
2. Yves Colin de Verdière, *Un principe variationnel pour les empilements des cercles*, Invent. Math. **104** (1991).
3. M. D. Kovalev, *On curvatures of tangent spheres*, Trudy Mat. Inst. Steklov.; in Russian **196** (1991), 90–92.
4. Greg Kuperberg and Oded Schramm, *Average kissing numbers for non-congruent sphere packings*, Math. Res. Lett. **1**; no. 3 (1994).
5. Igor Rivin, *Euclidean structures on simplicial surfaces and hyperbolic volume*, Ann. of Math. **139**; no. 2 (1994).
6. ———, *Some further remarks on combinatorial scalar curvature*, preprint (1995).

Mathematics Department, University California at Santa Barbara, Santa Barbara, CA 93106

E-mail address: cooper@math.ucsb.edu

Mathematics Institute, Warwick University, Coventry

Current address: Mathematics Department, California Institute of Technology, Pasadena, CA 91125

E-mail address: rivin@cco.caltech.edu