THE A-POLYNOMIAL HAS ONES IN THE CORNERS

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1. *Definition of the A-polynomial*

The A-polynomial was introduced in [**3**] (see also [**5**]), and we present an alternative definition here. Let M be a compact 3-manifold with boundary a torus T. Pick a basis λ, μ of $\pi_1 T$, which we shall refer to as the longitude and meridian. Consider the subset R_U of the affine algebraic variety $R = \mathrm{Hom}(\pi_1 M, \mathrm{SL}_2 \mathbb{C})$ having the property that $\rho(\lambda)$ and $\rho(\mu)$ are upper triangular. This is an algebraic subset of R, since one just adds equations stating that the bottom-left entries in certain matrices are zero. There is a well-defined eigenvalue map

$$\xi \equiv (\xi_\lambda \times \xi_\mu) \colon R_U \longrightarrow \mathbb{C}^2$$

given by taking the top-left entries of $\rho(\lambda)$ and $\rho(\mu)$. Thus the closure of the image $\xi(C)$ of an algebraic component C of R_U is an algebraic subset of $\mathbb{C}^2$. In the case that the image closure is a curve, there is a polynomial, unique up to constant multiples, which defines this curve [**7**, (1.13)]. The product over all components of R_U having this property of the defining polynomials for these curves is the A-polynomial. It is shown in [**3**] that the constant multiple may be chosen so that the coefficients are integers. The additional requirement that there is no integer factor of the result means that the A-polynomial is defined up to sign.

The main new result in this paper is that the coefficients of the A-polynomial which appear in the corners of the Newton polygon of A are all ± 1. We use this to give another proof that the roots of the edge polynomials of the A-polynomial are roots of unity. Proofs of this fact may be found in [**3**] and [**4**]. We also obtain as a corollary that the trace of the core curve under a Dehn filling of a one-cusped manifold, along a non-boundary slope, is an algebraic integer. Equivalently, the eigenvalue is an algebraic unit. We note that Bass has shown that if there are no closed incompressible surfaces, then all traces are algebraic integers. Our proof shows that certain traces are forced to be algebraic integers even in the presence of a closed incompressible surface. Our method is to obtain information about the A-polynomial using $\mathscr{P}$-adic valuations associated to representations all of whose traces are algebraic. These valuations determine an action on a tree via the Tits–Bass–Serre theory.

2. *$\mathscr{P}$-adic valuations*

When one has a field F and a discrete rank-1 valuation v on F, the Tits–Bass–Serre theory provides an action of $\mathrm{SL}_2 F$ on a simplicial tree. This fact has been exploited

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in 3-manifolds by Culler and Shalen in the case that F is a function field for a complex curve using valuations which are basically 'polynomial degree', and by Bass in the case that F is a number field using $\mathscr{P}$-adic valuations. The A-polynomial is derived from a curve of representations, and the relation between the Newton polygon of the A-polynomial and boundary slopes of the knot complement uses the valuation studied by Culler and Shalen. Our goal is to discover what information about the A-polynomial can be obtained by using $\mathscr{P}$-adic valuations. To see that this makes sense, consider a curve C of representations of the torally bounded 3-manifold M. Let us suppose that C projects into a curve (minus finitely many points) under the eigenvalue map. There are points on this curve where one of the eigenvalues is algebraic over $\mathbb{Q}$, and a little thought shows that this means that any representation $\rho\colon \pi_1 M \to \mathrm{SL}_2\mathbb{C}$ on C which projects to this eigenvalue can be conjugated so that it has image in $\mathrm{SL}_2 F$ for some number field F. Now it is well known that a valuation on a field extends to a finite extension of the field. Thus the p-adic valuation on $\mathbb{Q}$ can be extended over F. This leads to splittings of $\pi_1 M$, and a natural question is to ask whether one obtains new information from these valuations. In particular, might they detect incompressible surfaces which are not detected by the valuations previously considered? It is a consequence of our main result that these valuations give no new boundary slopes.

Suppose that

$$f(x,y) = \sum_{i,j} c_{a,b}\, x^a y^b$$

is a polynomial in two variables. Then we define the Newton polygon, $\mathrm{Newt}(f)$, of $f(x,y)$ to be the convex hull in the plane of the set $\{(a,b) \mid c_{a,b} \neq 0\}$. (This terminology is not entirely standard; compare [**2**].)

The main result of this analysis is Theorem 2.5, which says that the coefficients of the A-polynomial in the corners of the Newton polygon are all ± 1. This is combined with information obtained from the volume-form to give yet another proof of the fact presented in [**3**] and [**4**], namely that the limiting eigenvalues associated to a degeneration are roots of unity. However, this approach does not yield the connection between the order of the root of unity and the number of boundary curves on a splitting surface. Our excuse for doing this is that this proof is simpler than any of the other proofs. We wonder if there may be other consequences of Theorem 2.5.

We first recall some general facts. Let k be a number field, that is, a finite extension of $\mathbb{Q}$. Denote by $\mathscr{O}_k$ the ring of algebraic integers of k. For each prime $\mathscr{P}$ of k, we denote by $v_{\mathscr{P}}$ the discrete valuation (the *$\mathscr{P}$-adic valuation*) associated to $\mathscr{P}$. Let $k_{\mathscr{P}}$ denote the completion of k with respect to the valuation $v_{\mathscr{P}}$, and $\mathscr{O}_{\mathscr{P}}$ the ring of $\mathscr{P}$-adic integers, that is, the set $\{x \in k_{\mathscr{P}} \colon v_{\mathscr{P}}(x) \geqslant 0\}$. The ring $\mathscr{O}_{\mathscr{P}}$ has a unique maximal ideal generated by an element π, called *a local uniformising parameter*, such that $v_{\mathscr{P}}(\pi) = 1$. It follows from number theory that $\mathscr{O}_{\mathscr{P}}/\pi\mathscr{O}_{\mathscr{P}}$ is a finite field (*the residue class field*) of characteristic p, where $\mathscr{P} \mid p$. Let P be the set of all $\mathscr{P}$-adic valuations associated to k. For our purposes, the key lemma is as follows is (compare [**1**, Lemma 6.8.2]).

LEMMA 2.1. *If k is a number field, then $\mathscr{O}_k = \bigcap_{v_{\mathscr{P}} \in P} \mathscr{O}_{\mathscr{P}}$.*

In words, this means that we can characterise the algebraic integers as those elements which value non-negatively in every valuation on the number field. This has the following consequence.

LEMMA 2.2. *Suppose that p is a prime number and $r > 0$, and*

$$h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_k p^r t^k$$

is an integer polynomial, irreducible over $\mathbb{Z}$, with the property that c_k is prime to p and not ± 1. Let b be some root of $h(t)$.

Then there is a valuation v on $\mathbb{Q}(b)$ so that $v(p+1/p) \geqslant 0$ and $v(b) < 0$.

Proof. Multiply $h(t)$ by p^{nk-r} for some large positive integer n. The highest coefficient of the resulting polynomial is $c_k p^{nk} t^k = c_k (p^n t)^k$, so if we set $d = p^n b$, we see that d satisfies the polynomial

$$h^*(t) = c_0 p^{nk-r} + \ldots + c_{k-1} p^{n-r} t^{k-1} + c_k t^k.$$

This polynomial is still $\mathbb{Z}$ irreducible, since d and b visibly define the same field over $\mathbb{Q}$. In particular, both b and d are not algebraic integers.

Let v_p be the p-adic valuation on $\mathbb{Q}$, so that $v_p(p^n) = n$, and let $\tilde{v}_p$ be any extension of this valuation to $\mathbb{Q}(d) = \mathbb{Q}(b)$. We claim that $\tilde{v}_p(d) \geqslant 0$. The reason is that if this were not so, then, since p does not divide c_k, $\tilde{v}_p(c_k) = 0$, so we have $\tilde{v}_p(c_k d^k) = \tilde{v}_p(c_k) + k\tilde{v}_p(d) = k\tilde{v}_p(d)$.

However, all of the other terms in the sum defining $h^*(d)$ are of the shape $x_j d^j$, where $j < k$ and x_j is an integer, so that, since $\tilde{v}_p(d) < 0$, we have

$$\tilde{v}_p(x_j d^j) = \tilde{v}_p(x_j) + j\tilde{v}_p(d) \geqslant j\tilde{v}_p(d) > k\tilde{v}_p(d),$$

and we see that $\tilde{v}_p(c_k d^k)$ is the unique term of minimal valuation. However, this implies that

$$\tilde{v}_p(0) = \tilde{v}_p(h^*(d)) = \tilde{v}_p(c_k d^k),$$

a contradiction.

Now, by Lemma 2.1, there is some valuation v on $\mathbb{Q}(d)$ for which $v(d) < 0$, and the above paragraph shows that this cannot be the p-adic valuation when restricted to $\mathbb{Q}$. The only valuations on $\mathbb{Q}$ are the q-adic valuations. Hence for some $q \neq p$, we have $v_q(d) < 0$ and $v_q(p+1/p) = v_q(p^2+1) \geqslant 0$, and $v_q(p)$ and $v_q(1/p)$ are both zero.

Moreover, $0 > v_q(d) = v_q(p^n b) = nv_q(p) + v_q(b) = v_q(b)$, as was required.

Suppose that we choose a new basis α, β on $\pi_1 T$. This changes the Newton polygon of the A-polynomial by a shear, but coefficients are not changed. Given a corner of Newt(A), choose new coordinates α, β on $\pi_1 T$ so that α is not a boundary slope and no two terms have the same L-degree. We can also arrange that the given corner is the unique term in $A(M, L)$ of maximal L-degree. It is shown in [**3**] that the slope of a side of the Newton polygon for A is always a boundary slope.

Assume that we have arranged the greatest common divisor of the coefficients to be 1, and that term of highest L-degree is $mM^s L^r$. Let C' be a component of the representation variety which contributes a factor to the A-polynomial. If the dimension of C is bigger than 1, then we may intersect it with hyperplanes defined by integral equations, to produce a curve C defined by equations with integer coefficients and such that $\xi(C')$ and $\xi(C)$ have the same closure. We have a map $C \to \mathbb{C}$ given by mapping ρ to $\operatorname{tr}(\rho(\alpha))$. We may assume that the trace of $\rho(\alpha)$ is not constant on the curve C, since if it were, it follows from the fact that α is not a boundary slope that the traces of all peripheral elements are constant on C. This would imply that C does not contribute to the A-polynomial.

The map ξ is therefore dominating and can omit only finitely many values. We are now going to specialise M to be a prime p, which we choose so that:

- $p+1/p$ is in the image of ξ; and
- p is coprime to m.

LEMMA 2.3. *If we factorise $A(p, L)$ over the integers as $\ell \prod_{i=1}^{K} h_i(L)$, then $m \mid \ell$.*

Proof. If not, then since m is prime to p, we can find some polynomial $h_j(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_k p^r t^k$ for which the highest common factor of m and c_k is larger than 1. By Lemma 2.2, there is a valuation on the field $\mathbb{Q}(b)$ with (i) $v(p+1/p) \geqslant 0$, and (ii) $v(b) < 0$. By hypothesis, there is a representation ρ on the curve C with $\operatorname{tr}(\rho(\alpha)) = p + 1/p$ and $\operatorname{tr}(\rho(\beta)) = b + 1/b$. Furthermore, the set of such representations in C is finite. Thus ρ is an isolated point in a variety defined by polynomial equations with integer coefficients. By Lemma 2.4 below, the image of this representation is in $\mathrm{SL}_2 K$ for some number field K. Extend the valuation v to a valuation on K; Bass–Serre theory applied to $\mathrm{SL}_2 K$ now gives a splitting of the knot group, which is non-trivial since the trace of the element β has negative valuation (see [**6**, Theorem 2.1.2]). Moreover, the element α can be conjugated into a vertex stabiliser, since its trace has positive valuation. It is shown in [**6**] that this makes α into a boundary slope, a contradiction.

LEMMA 2.4. *Let $\mathbb{Z}[X_1, X_2, \ldots, X_k]$ be the ring of polynomials with integer coefficients in the variables $X_1, \ldots, X_k$. Let $p_1, p_2, \ldots, p_n \in \mathbb{Z}[X_1, X_2, \ldots, X_k]$, let I be the ideal generated by these polynomials, and let S be the affine algebraic set determined by I. Suppose that S has a component V which is a single point. Then the coordinates of V are algebraic over $\mathbb{Z}$.*

Proof. Let $x_1, x_2, \ldots, x_k \in \mathbb{C}$ be the coordinates of V, and let $K = \mathbb{Q}(x_1, x_2, \ldots, x_k)$; we must show K is an algebraic extension of $\mathbb{Q}$. If not, then we may suppose that $x_1, \ldots, x_r$ are independent transcendentals, and that $x_{r+1}, \ldots, x_k$ are algebraic over $K_1 = \mathbb{Q}(x_1, \ldots, x_r)$. Let $y_1, \ldots, y_r$ be transcendentals so that $|x_i - y_i|$ is small. Thus there is a field isomorphism $f\colon \mathbb{Q}(x_1, \ldots, x_r) \to \mathbb{Q}(y_1, \ldots, y_r)$. We claim that f extends to an isomorphism defined on K with image some subfield of $\mathbb{C}$ and such that $|f x_i - x_i|$ is small for all $1 \leqslant i \leqslant k$. To see this, since x_{r+1} is algebraic over K_1 it has minimum polynomial $m(t)$ over K_1 with coefficients in K_1. Let $m'(t)$ be the polynomial obtained by applying f to the coefficients of $m(t)$. Then $m'(t)$ is close to $m(t)$, so there is a complex number z which is a zero of $m'(t)$ that is near x_{r+1}. Then define $f(x_{r+1}) = z$. Continue this way. Thus $f(V)$ is very close to V. Since the polynomials defining S have integer coefficients, the algebraic set they define is invariant under f, so V is not isolated, a contradiction.

Our main theorem in this section now follows.

THEOREM 2.5. *If $A(M, L)$ is normalised so that the greatest common divisor of the coefficients is* 1, *then the corner vertices of the Newton polygon of A are* ± 1.

Proof. In the notation already established, this amounts to showing that $m = \pm 1$. With our choice of coordinates, we may write

$$A(M, L) = \sum_{i=0}^{r-1} k_{j_i} M^{j_i} L^i + m M^s L^r,$$

and we specialise to $M = p$, then obtain some polynomial all of whose coefficients are divisible by ℓ. By Lemma 2.3, $m \mid \ell$, and it follows that m divides every coefficient $k_{j_i} p^{j_i}$, and hence, since it is prime to p, we have $m \mid k_{j_i}$ for every i. Our normalisation gives the GCD of these integers as ± 1, proving the result.

REMARK. We note that since the ring of algebraic integers is integrally closed, it follows easily that ξ is an algebraic unit if and only if $\xi + \xi^{-1}$ is an algebraic integer.

Bass showed in [**1**] that if a closed hyperbolic 3-manifold contains no closed incompressible surface, then the trace of every element is an algebraic integer. In the light of the remark, the next result implies the following. If a closed hyperbolic 3-manifold is obtained by Dehn filling a one-cusped manifold along a non-boundary slope, then the trace of the core curve after Dehn filling is an algebraic integer.

THEOREM 2.6. *Let N be a compact* 3*-manifold with boundary a torus. Suppose that α is an essential simple closed curve on this torus which is not a boundary slope. Let $N(\alpha)$ be the result of Dehn filling along α. Let ρ be any irreducible representation of $\pi_1(N(\alpha))$ into* $\mathrm{SL}_2\mathbb{C}$, *such that ρ is non-trivial on the boundary torus. Let ξ be the eigenvalue of the core curve γ of the attached solid torus. Then ξ is an algebraic unit.*

Proof. Suppose that $\alpha = \mu^p \lambda^q$, with p, q coprime (not both zero), is a simple closed curve on ∂N, with slope p/q, and let V be a solid torus. Dehn filling along α gives the manifold $N(\alpha)$ obtained by identifying the boundary ∂V with ∂N so that the boundary of a meridian disc of V is identified to α. A representation $\rho \colon \pi_1(N) \to \mathrm{SL}_2\mathbb{C}$ factors through $\pi_1(N(\alpha))$ if and only if $\rho(\alpha)$ is the identity. Suppose that

$$\rho_0 \colon \pi_1(N(\alpha)) \longrightarrow \mathrm{SL}_2\mathbb{C}$$

is an irreducible representation. The inclusion of N into $N(\alpha)$ induces an epimorphism of $\pi_1(N)$ onto $\pi_1(N(\alpha))$, so we have an irreducible representation ρ_0' of $\pi_1(N)$ into $\mathrm{SL}_2\mathbb{C}$. An argument due to Thurston [**9**, (5.6.1), (5.6.2)] shows that this representation can be deformed. In fact, there is a curve C of representations of $\pi_1(N)$ containing ρ_0' and a loop in ∂N whose trace is non-constant as ρ varies over C. Thus C determines a component of the algebraic set $A(M, L) = 0$, and ρ_0' determines a point x on this curve which also lies on the curve $M^p L^q = 1$. We introduce a complex parameter T for this latter curve, so

$$M = T^{-q}, \quad L = T^p.$$

The core curve of V is homotopic to $\gamma = \mu^r \lambda^s$ with $ps - qr = 1$, and so this parameter may be interpreted as an eigenvalue ξ of $\rho(\gamma)$, since

$$\xi = M^r L^s = T^{-qr+ps} = T.$$

It follows that the minimum polynomial $g_\xi(T)$ over $\mathbb{Q}$ of ξ divides $f(T) \equiv A(T^{-q}, T^p)$. The coefficient c_n of T^n in $f(T)$ is a sum of coefficients of terms in $A(M, L)$, namely

$$A(M,L) = \sum a_{ij} L^i M^j \quad \text{and} \quad c_n = \sum_{pi-qj=n} a_{ij}.$$

Let n be the degree of $f(T)$, and suppose that there is more than one term in the A-polynomial contributing to the coefficient of T^n in $f(T)$, say a_{ij} and a_{kl} such that

$$pi - qj = n = pk - ql.$$

Then there is an edge of the Newton polygon Newt(A) of A containing the points (i,j) and (k,l). The slope of this edge is p/q, and from [**3**] this implies that p/q is the boundary slope of some essential surface in N. Thus α is a boundary slope.

Thus if α is not a boundary slope, then c_n equals the coefficient of $A(M, L)$ in one of the corners of the Newton polygon, and by Theorem 2.5 this coefficient is ± 1. Hence $f(T)$ is monic. Now it is easy to see (and shown in [**3**]) that $A(1/M, 1/L) = A(M, L)$, and so $f(T)$ is a reciprocal polynomial. Thus the coefficient of the lowest term in f is also ± 1. Since $g_\xi(T)$ divides $f(T)$, it follows that the highest and lowest coefficients of $g_\xi(T)$ are ± 1. Thus ξ is an algebraic unit.

We now give a simple argument using the volume form to show that edge polynomials have roots on the unit circle.

LEMMA 2.7. *Every root of every edge polynomial of the A-polynomial of a knot lies on the unit circle.*

Proof. Let C be an affine curve in $\mathbb{C}^2$. Then a *Puiseux* parametrisation at a point (x_0, y_0) is a function $\mathbb{C} \to C$ given by

$$x(t) = x_0 + t^p, \quad y(t) = y_0 + \sum_{n=0}^{\infty} a_n t^n.$$

Every point on C has a neighbourhood which is covered by the images of finitely many such parametrisations, corresponding to the different sheets of C passing through (x_0, y_0). Given a basis α, β of $\pi_1 T$, we use the same symbols to indicate the coordinates of a point on C using this basis. We study a particular degeneration, and assume that the basis is chosen so that $\beta \to K$ and $\alpha \to 0$. The sequence of representations for this degeneration eventually lies in the image of a Puiseux parametrisation:

$$\alpha(t) = t^p, \quad \beta(t) = K + \sum_{n=0}^{\infty} a_n t^n.$$

A representation of $\pi_1 M$ into $\mathrm{SL}_2\mathbb{C}$ determines a real number called the *volume* of the representation, and a formula due to Hodgson gives

$$\pm 2dV = \ell_\alpha d\theta_\beta - \ell_\beta d\theta_\alpha;$$

see [**3**] for details. The integral of dV around the loop parametrised as $t = e^{r+i\theta}$, where r is large negative and $0 \leqslant \theta \leqslant 2\pi$, must give 0. We now compute this:

$$\ell_\alpha = \log|\alpha| = pr, \quad \ell_\beta = \log|\beta| = \log|K| + O(e^r), \quad \theta_\alpha = p\theta, \quad \theta_\beta = \operatorname{Im}\log\beta.$$

Differentiating gives

$$\frac{d\theta_\beta}{d\theta} = \mathrm{Im}\left(\frac{1}{\beta}\frac{d\beta}{d\theta}\right) = \mathrm{Im}\left(\frac{1}{\beta}\sum_{n=0}^{\infty} nia_n e^{n(r+i\theta)}\right) = O(e^r).$$

Hence

$$2\frac{dV}{d\theta} = \ell_\alpha \frac{d\theta_\beta}{d\theta} - \ell_\beta \frac{d\theta_\alpha}{d\theta} = p\log|K| + O(e^r).$$

Integrating this yields

$$0 = \int_0^{2\pi} \frac{dV}{d\theta}\,d\theta = \int_0^{2\pi} -p\log|K|\,d\theta + O(e^r) = -2\pi\log|K| + O(e^r).$$

Thus $\log|K| = 0$, so $|K| = 1$ as claimed.

The following is well known.

Lemma 2.8. *Suppose that the complex number z is algebraic over $\mathbb{Q}$ with minimum polynomial $p \in \mathbb{Z}[t]$, and suppose that all the roots of p lie on the unit circle. Suppose also that p is monic. Then z is a root of unity.*

Proof. Write $p = t^n + a_1 t^{n-1} + \ldots + a_n$. Then the a_i are the symmetric functions of the roots of p. Since these roots all lie on the unit circle, $|a_i|$ is bounded above by a binomial coefficient. The set $\mathscr{S}$ of monic integer polynomials with coefficients so bounded is finite. Let p_m be the minimum polynomial of z^m for $m > 0$. The other roots of p_m are a subset of the mth powers of the roots of p. Thus they all lie on the unit circle. Thus p_m divides an element of $\mathscr{S}$. Since there are only finitely many such divisors, it follows that there are only finitely many possible values for z^m as m varies. Thus z is a root of unity.

Theorem 2.9 (see [**3**] and [**4**]). *Every root of every edge polynomial of the A-polynomial of a knot is a root of unity.*

Proof. By Theorem 2.5 every edge polynomial is monic, and by Lemma 2.7 every root of these polynomials lies on the unit circle. But now Lemma 2.8 gives that the edge polynomials have all roots on the unit circle.

Corollary 2.10. *The p-adic valuation v_p does not detect any boundary slopes other than those given by slopes of edges of the Newton polygon.*

Proof. The precise statement is as follows. Suppose that

$$\rho\colon \pi_1 M \longrightarrow \mathrm{SL}_2\,\mathbb{F},$$

where $\mathbb{F}$ is a number field. Suppose that $p \in \mathbb{Z}$ is a prime, and $v_{\mathscr{P}}$ is an extension of the p-adic valuation to $\mathbb{F}$. Suppose that this valuation on $\mathrm{SL}_2\,\mathbb{F}$ gives rise to an action of $\pi_1 M$ on a simplicial tree with no fixed point. Then this gives rise to an incompressible surface in M with non-empty boundary provided that $v_{\mathscr{P}}(\mathrm{tr}\,\rho\alpha) < 0$ for some α in $\pi_1 T$. A primitive element $\beta = \mu^a\lambda^b$ of $\pi_1 T$ with $v_{\mathscr{P}}(\mathrm{tr}\,\rho\beta) \geqslant 0$ is a boundary slope of M. Then the slope of β on T is a/b, which equals the slope of some side of $\mathrm{Newt}\,(A)$.

Let $L, M \in \mathbb{F}$ be the eigenvalues of the longitude and meridian for ρ, thus

$$0 = A(M, L) = \sum a_{ij} L^i M^j.$$

Now $v_{\mathscr{P}}(x+y) = \min\{v_{\mathscr{P}} x, v_{\mathscr{P}} y\}$ unless $v_{\mathscr{P}} x = v_{\mathscr{P}} y$; also, $v_{\mathscr{P}} 0 = \infty$. It follows that the subset of $\{a_{ij} L^i M^j\}$ consisting of those terms with minimal value has at least two elements. Now

$$v_{\mathscr{P}}(a_{ij} L^i M^j) = v_{\mathscr{P}}(a_{ij}) + i v_{\mathscr{P}}(L) + j v_{\mathscr{P}}(M),$$

and since $a_{ij} \in \mathbb{Z}$ we have $v_{\mathscr{P}}(a_{ij}) \geqslant 0$. The function

$$f\colon \mathbb{Z}^2 \cap \mathrm{Newt}(A) \longrightarrow \mathbb{R}$$

given by

$$f(i,j) = i v_{\mathscr{P}}(L) + j v_{\mathscr{P}}(M)$$

takes its minimal values on the boundary of Newt (A). Suppose that there is a single point where f takes on its minimum. Then this is a corner of Newt (A). Since the coefficient of a corner term is ± 1, the value of this corner term is given by f at that corner. Since

$$v_{\mathscr{P}}(a_{ij} L^i M^j) = v_{\mathscr{P}}(a_{ij}) + f(i,j) \geqslant f(i,j),$$

this corner term is the only term in $A(M, L)$ of minimal value which is impossible. Thus there is an entire side of Newt (A) on which f takes its minimal values. If the terms at the corners of Newt (A) which are the ends of this side are $\pm L^a M^b$ and $\pm L^c M^d$, then these terms have the same $v_{\mathscr{P}}$-value. Thus $L^{a-c} M^{b-d}$ has value zero, and thus $\lambda^{a-c}\mu^{b-d}$ is the element in $\pi_1 T$ which stabilises some edge of the tree associated to $v_{\mathscr{P}}$, and therefore this element is the boundary slope. The slope of $\lambda^{a-c}\mu^{b-d}$ is $\dfrac{b-d}{a-c}$, but this is just the slope of the edge of Newt (A) considered.

References

1. H. Bass, 'Groups of integral representation type', *Pacific J. Math.* 86 (1980) 15–51.
2. E. Brieskorn and H. Knörrer, *Plane algebraic curves* (Birkhäuser, Heidelberg, 1986).
3. D. Cooper, M. Culler, H. Gillet, D. D. Long and P. B. Shalen, 'Plane curves associated to character varieties of knot complements', *Invent. Math.* 118 (1994) 47–84.
4. D. Cooper and D. D. Long, 'Roots of unity and the character variety of a knot complement', *J. Austral. Math. Soc.* 55 (1993) 90–99.
5. D. Cooper and D. D. Long, 'Remarks on the A-polynomial of a knot', *J. Knot Theory Ramifications*, to appear.
6. M. Culler and P. B. Shalen, 'Varieties of group representations and splittings of 3-manifolds', *Ann. of Math.* 117 (1983) 109–145.
7. R. Hartshorne, *Algebraic geometry* (Springer, Berlin, 1977).
8. D. A. Markus, *Number fields* (Springer, New York, 1977).
9. W. P. Thurston, 'The geometry and topology of 3-manifolds', preprint.

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