TRIANGULATING 3-MANIFOLDS USING 5 VERTEX LINK TYPES

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We show that a closed orientable 3-manifold can be triangulated in a simple way locally. There are 5 triangulations of $S^2$ with the property that every such manifold has a triangulation in which the link of each vertex is one of these 5 link types. This triangulation is obtained from a paving of the manifold by cubes. In this paving, the order of every edge is 3, 4 or 5. Denoting the union of the edges of order 3 (respectively 5) by $\sum_3$ (respectively $\sum_5$), then $\sum_3$ and $\sum_5$ are disjoint 1-submanifolds. It is known that for any dimension $n$, there is a finite set of link types such that every $n$-manifold has a triangulation in which the link of each vertex is in this set. However for $n>3$, no such set is known.

If $K$ is a simplicial complex, we denote the barycentric subdivision of $K$ by $K'$, and the suspension of $K$ by $\Sigma K$. The set $J$ consists of the 5 triangulations of $S^2$ listed below:

(T1) $\partial$(octahedron)
(T2) $[\partial$(3-simplex)]'
(T3) $[\sum\partial$(triangle)]'
(T4) $[\sum\partial$(square)]' = $[\partial$(octahedron)]'
(T5) $[\sum\partial$(pentagon)]'

Each of these triangulations is obtained by doubling along the boundary a suitable triangulation of a 2-disc. These triangulations of the 2-disc are shown in Fig. 1.

Theorem 1. Let $M$ be a closed, orientable 3-manifold. Then $M$ can be triangulated so that the link of every vertex is in $J$.

A paving of a manifold by cubes is like a triangulation, but made out of n-dimensional cubes instead of simplexes. In particular two cubes either meet in a face (which is a cube of lower dimension) or are disjoint. Suppose a 3-manifold $M$ is paved by cubes, and $e$ is an edge of a 3-cube $C$, then the order of $e$ is $\text{card}\{C': C' \text{ is a 3-cube in } M \text{ and } e \subset C'\}$.

Theorem 2. Let $M$ be a closed, orientable 3-manifold. Then there is a paving of $M$ by cubes such that the order of every edge is 3, 4 or 5. Furthermore $\sum_3$ and $\sum_5$ are disjoint 1-submanifolds.

Proof of Theorem 2. According to [1] every closed orientable 3-manifold is obtained by taking a suitable covering of $S^3$ branched over the Borromean rings. Let $B \subset S^3$ denote the Borromean rings. Now the 3-torus $T^3 = S^1 \times S^1 \times S^1$ is obtained by doing 0-framed surgery along each component of $B$. Thus there is a link $L$ of three components in $T$ with $T^3 - L = S^3 - B$. The components of $L$ can be chosen to be $S^1 \times (\theta_1, \theta_1)$, $S^1 \times (\theta_2, \theta_2)$ and $(\theta_3, \theta_3) \times S^1$. The order of each edge is 3, 4 or 5, and $\sum_3$ and $\sum_5$ are disjoint 1-submanifolds.
where \( \theta_i \in S^1 \) and all the \( \theta_i \)'s are distinct. \( T^3 \) can be represented as the unit cube, \( A \), in \( \mathbb{R}^3 \) with opposite faces identified. Then \( L \cap A \) consists of three disjoint arcs labelled \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) which are parallel to the coordinate axes \( x_1, x_2 \) and \( x_3 \) respectively; see Fig. 2. Let \( P_A \) be the paving of \( A \) by \( N^3 \) cubes each of side length \( N^{-1} \) (where \( N \) is suitably large, e.g. \( N = 10 \)). The corresponding paving of \( T^3 \) obtained by identifying opposite faces of \( A \) is regular, i.e. every edge has order 4. We may assume that \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are disjoint from the 1-skeleton of \( P_A \). For \( i = 1, 2, 3 \) let \( N_i \) be the union of the cubes of \( P_A \) which intersect \( \gamma_i \). We may assume that \( N_1, N_2 \) and \( N_3 \) are disjoint. In \( T^3 \), each \( N_i \) glues up to become a solid torus \( X_i \), and \( X_i \) is paved with \( N \) cubes. This paving of \( X_i \) is the product paving of \( D^2 \) paved as a single square, and \( S^1 \) paved by \( N \) intervals. Let \( D = T^3 - (X_1 \cup X_2 \cup X_3) \). Then \( \partial D \) consists of three tori: \( T_1, T_2 \) and \( T_3 \). Each \( T_i \) is paved in an identical way by \( 4N \) squares. This paving is the product of a paving of \( S^1 \) by four intervals with a paving of \( S^1 \) by \( N \) intervals. Now pave \( T^2 \times I \) using the product of the paving on \( T^2 \) given by the paving of any \( T_i \), and the paving of \( I \) as a single interval. Take three such paved copies of \( T^2 \times I \) and glue one copy along \( T^2 \times 0 \) onto each of \( T_1, T_2 \) and \( T_3 \) so that
the pavings match up along the glueing. Topologically all we have done is to add a collar onto each boundary component of $D$. Call the resulting manifold $E$, and the paving of $E$, $P_E$. Then $\sum_S(P_E)$ consists of 12 $S^1$'s, four coming from each of $T_1$, $T_2$ and $T_3$. Each of these $S^1$'s on $T_i$ runs parallel to $\gamma_i$. $\sum_S(P_E) = \emptyset$.

Let $p: M \rightarrow S^3$ be a covering of $S^3$ branched over $B$. Then identifying $S^3 - B$ with $E$ gives a paving $\bar{P}_E$ of $p^{-1}(E)$ by lifting the paving $P_E$. Locally the pavings $P_E$ and $\bar{P}_E$ are the same. To obtain a paving of $M$, we must attach paved solid tori to each boundary component of $p^{-1}(E)$. Let $U$ be a boundary component contained in $p^{-1}(T_i)$, then the paving of $U$ is obtained by taking some abelian cover of the paving of $T_i$. The framing of the solid torus $Y$ to be glued to $U$ is determined by $\beta_u = U \cap p^{-1}(\gamma_i)$ ($\gamma_i$ may be regarded as a simple closed curve on $T_i$). Thus $U$ is paved as a product of a paving of $\beta_u = S^1$ by $N_p$ intervals and a paving of $S^1$ by $4q$ intervals where the positive integers $p$, $q$ depend on the abelian cover $U \rightarrow p(U)$. Now choose $N = 4M$, $M$ an integer. Then pave $Y$ as $(S^1$ paved by $4q$ intervals) $\times$ $(D^1$ paved by $M_p$ intervals) $\times$ $(D^1$ paved by $M_p$ intervals). Then $Y$ may be glued onto $U$ so that the pavings match up. This introduces four components to $\sum_S$, each component is a circle on $U$ which projects down to a meridian of $\gamma_i$ in $T^3$. Doing this for each boundary component $U$ gives the required paving of $M$. □

**Remark.** This paving puts a Euclidean cone-manifold structure on $M$, in which the singular locus is a link, and the cone angles are $3\pi/2$ and $5\pi/2$.

**Proof of Theorem 1.** The cube may be triangulated as $[(3\text{-simplex})]$, see Fig. 1(T2). Each face of the cube is triangulated as cone($\delta$ square). Thus a triangulation of $M$ is obtained from a paving by cubes using this triangulation for each cube. The link of a vertex at the centre of a cube is $(T2)$. The link of a vertex at the centre of a face of a cube is $(T1)$. The link of a vertex $v$ at a corner of a cube is $(T3)$ if $v$ lies on $\sum_S$, is $(T5)$ if $v$ lies on $\sum_S$, and is $(T4)$ otherwise. □

**Note added in Proof**

Kevin Walker has shown independently, using a variant of our method, that 3 vertex link types suffice.

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