THE VOLUME OF A CLOSED HYPERBOLIC 3-MANIFOLD
IS BOUNDED BY $\pi$ TIMES THE LENGTH OF ANY
PRESENTATION OF ITS FUNDAMENTAL GROUP

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Theorem 0.1. Suppose $M$ is a closed hyperbolic 3-manifold. Given a presentation of $\pi_1 M$ let $L$ be the sum of the word-lengths of the relations and $n$ the number of relations of length at least 3. Then $\text{volume}(M) < \pi(L - 2n)$.

Proof. A pleated disc is a map covered by a map of a disc into $\mathbb{H}^3$ such that there is a triangulation of the disc with vertices only on the boundary of the disc and with the property that the image of each 2-simplex is a geodesic 2-simplex in $\mathbb{H}^3$.

A presentation of $M$ gives a set of generators and relations. For simplicity, we will assume every relation has word length at least 3. This may be realized geometrically by a map $f : S \to M$ of a 2-complex $S$ which induces an isomorphism of $\pi_1 S$ onto $\pi_1 M$. The map $f$ may be homotoped so that edges map to geodesics and $f$ restricted to each 2-cell is a pleated disc. The area of a pleated disc is at most $\pi$ times the number of 2-simplices. The boundary of a 2-cell, $D$, in $S$ represents a relation, and the number of 2-simplices in $D$ is the number of edges in $\partial D$ minus 2. The number of edges in $\partial D$ is the word length of the relation represented by $\partial D$. Thus the total surface area of $f(S)$ is at most $\pi(L - 2n)$.

Let $X$ be the closure of a component of $M - f(S)$; then $X$ lifts to $\mathbb{H}^3$. For otherwise, there is a loop $\gamma$ in $X$ which is not contractible in $M$. Since $S$ is mapped into $M - \gamma$, the isomorphism $f_* : \pi_1 S \to \pi_1 M$ factors through $\pi_1(M - \gamma)$. Thus the composite

$$\pi_1 M \cong \pi_1 S \to \pi_1(M - \gamma) \to \pi_1 M$$

is the identity, where the second map is induced by inclusion. Now $M$ is aspherical, hence $\pi_2(M - \gamma) = 0$ because otherwise, by the sphere theorem, $\gamma$ would be contained inside a ball and thus contractible in $M$. Hence $M - \gamma$ is a $K(\pi, 1)$ and thus the first homomorphism is induced by a continuous map $M \to (M - \gamma)$. Thus the composite

$$M \to (M - \gamma) \to M$$

is a $\pi_1$-isomorphism, hence a homotopy equivalence. Consideration of the induced map on $H_3$ gives a contradiction:

$$H_3(M) \to H_3(M - \gamma) \to H_3(M)$$

since the composite is an isomorphism and $M$ is closed.

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The isoperimetric inequality (for example [1], p. 283) for $\mathbb{H}^3$ states that, for a given volume, the smallest ratio of surface area divided by volume is attained by a sphere. Computation shows that this ratio is always greater than 2. This asymptotic ratio is attained by a horosphere. Thus the surface area of the polyhedron $X$ in $\mathbb{H}^3$ is at least 2 times its volume. Now $S$ may be subdivided so that each 2-cell appears in exactly two such polyhedra, thus the total surface area of $S$ is greater than 1 times the volume of $M$. Putting this together with the first part gives the result. □

In his thesis, Matt White [2] has obtained (a much deeper result) an explicit bound on the diameter of $M$ in terms of the sum of the lengths of the relations. He also extends these results to the finite volume case.

REFERENCES


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