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DEHN SURGERY AND NEGATIVELY CURVED 3-MANIFOLDS

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1. Introduction

Dehn surgery is perhaps the most common way of constructing 3-manifolds, and yet there remain some profound mysteries about its behaviour. For example, it is still not known whether there exists a 3-manifold which can be obtained from S^3 by surgery along an infinite number of distinct knots.[1] (See Problem 3.6 (D) of Kirby's list [9]). In this paper, we offer a partial solution to this problem, and exhibit many new results about Dehn surgery. The methods we employ make use of well-known constructions of negatively curved metrics on certain 3-manifolds.

We use the following standard terminology. A *slope* on a torus is the isotopy class of an unoriented essential simple closed curve. If s is a slope on a torus boundary component of a 3-manifold X, then $X(s)$ is defined to be the 3-manifold obtained by Dehn filling along s. More generally, if $s_1, \dots, s_n$ is a collection of slopes on distinct toral components of ∂X, then we write $X(s_1, \dots, s_n)$ for the manifold obtained by Dehn filling along each of these slopes.

We also abuse terminology in the standard way by saying that a compact orientable 3-manifold X, with ∂X a (possibly empty) union of tori, is *hyperbolic* if its interior has a complete finite volume hyperbolic

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[1]Since this paper was written, John Osoinach has constructed a family of 3-manifolds, each with infinitely many knot surgery descriptions [Ph.D. Thesis, University of Texas at Austin].

structure. If X is hyperbolic, we also say that the core of the filled-in solid torus in $X(s)$ is a *hyperbolic knot.*

Let X be a hyperbolic 3-manifold, and let $T_1, \dots, T_n$ be a collection of components of ∂X. Now, associated with each torus T_i, there is a cusp in $\mathrm{int}(X)$ homeomorphic to $T^2 \times [1, \infty)$. We may arrange that the n cusps are all disjoint. They lift to an infinite set of disjoint horoballs in $\mathbb{H}^3$. Expand these horoballs equivariantly until each horoball just touches some other. Then, the image under the projection map $\mathbb{H}^3 \to \mathrm{int}(X)$ of these horoballs is a *maximal horoball neighbourhood* of the cusps at $T_1, \dots, T_n$. When $n = 1$, this maximal horoball neighbourhood is unique. Let $\mathbb{R}^2_i$ be the boundary in $\mathbb{H}^3$ of one of these horoballs associated with T_i. Then $\mathbb{R}^2_i$ inherits a Euclidean metric from $\mathbb{H}^3$. A slope s_i on T_i determines a primitive element $[s_i] \in \pi_1(T_i)$, which is defined up to sign. This corresponds to a covering translation of $\mathbb{R}^2_i$, which is just a Euclidean translation. We say that s_i has *length* $l(s_i)$ given by the length of the associated translation vector. When $n > 1$, this may depend upon the choice of a maximal horoball neighbourhood of $T_1 \cup \dots \cup T_n$, but for $n = 1$, the length of s_1 is a topological invariant of the manifold X and the slope s_1, by Mostow Rigidity [3, Theorem C.5.4]. The concept of slope length is very relevant to Dehn surgery along hyperbolic knots, and plays a crucial rôle in this paper. Note that slope length is measured in the metric on X, not in any metric that $X(s_1, \dots, s_n)$ may happen to have. This notion of slope length arises in the following well-known theorem of Gromov and Thurston, the so-called '2π' theorem.

Theorem (Gromov, Thurston [4, Theorem 9]). *Let X be a compact orientable hyperbolic 3-manifold. Let $s_1, \dots, s_n$ be a collection of slopes on distinct components $T_1, \dots, T_n$ of ∂X. Suppose that there is a horoball neighbourhood of $T_1 \cup \dots \cup T_n$ on which each s_i has length greater than 2π. Then $X(s_1, \dots, s_n)$ has a complete finite volume Riemannian metric with all sectional curvatures negative.*

The following theorem, which is the main result of this paper, asserts that (roughly speaking) any given 3-manifold M can be constructed in this fashion in at most a finite number of ways.

Theorem 4.1. *Let M be a compact orientable 3-manifold, with ∂M a (possibly empty) union of tori. Let X be a hyperbolic manifold and let $s_1, \dots, s_n$ be a collection of slopes on n distinct tori $T_1, \dots, T_n$ in ∂X, such that $X(s_1, \dots, s_n)$ is homeomorphic to M. Suppose that there exists in* $\mathrm{int}(X)$ *a maximal horoball neighbourhood of $T_1 \cup \dots \cup T_n$ on* *which each slope s_i has length at least $2\pi + \epsilon$, for some $\epsilon > 0$. Then, for any given M and ϵ, there is only a finite number of possibilities (up to isometry) for X, n and $s_1, \ldots, s_n$.*

This is significant because 'almost all' closed orientable 3-manifolds are obtained by such a Dehn surgery. More precisely, any closed orientable 3-manifold is obtained by Dehn filling some hyperbolic 3-manifold X ([13] and [12]). After excluding at most 48 slopes from each component of ∂X, all remaining slopes have length more than 2π [4, Theorem 11].

Theorem 4.1 has the following corollary, which is a partial solution to Kirby's Problem 3.6 (D).

Corollary 4.5. *For a given closed orientable 3-manifold M, there is at most a finite number of hyperbolic knots K in S^3 and fractions p/q (in their lowest terms) such that M is obtained by p/q-Dehn surgery along K and $|q| > 22$.*

Dehn filling also arises naturally in the study of branched covers. Recall [15] that a branched cover of a 3-manifold Y over a link L is determined by a transitive representation $\rho\colon \pi_1(Y - L) \to S_r$, where S_r is the symmetric group on r elements. The stabiliser of one of these elements is a subgroup of $\pi_1(Y - L)$ which determines a cover X of $Y - \text{int}(\mathcal{N}(L))$. The branched cover is then obtained by Dehn filling each component P of ∂X that is a lift of some component of $\partial\mathcal{N}(L)$. The Dehn filling slope on P is the slope which the covering map sends to a multiple of a meridian slope on $\partial\mathcal{N}(L)$. This multiple is known as the *branching index* of P. Branched covers are a surprisingly general construction. For example, any closed orientable 3-manifold is a branched cover of S^3 over the figure-of-eight knot [7]. Thus, the following corollary to Theorem 4.1 is useful.

Corollary 4.8. *Let M be a compact orientable 3-manifold with ∂M a (possibly empty) union of tori, which is obtained as a branched cover of a compact orientable 3-manifold Y over a hyperbolic link L, via representation $\rho\colon \pi_1(Y - L) \to S_r$. Suppose that the branching index of every lift of every component of $\partial\mathcal{N}(L)$ is at least 7. Then, for a given M, there are only finitely many possibilities for Y, L, r and ρ.*

This paper is organised as follows. In Section 2, we establish lower bounds on slope length from topological information. In Section 3, we review the proof of the '2π' theorem and establish a 'controlled' version of the theorem, which involves estimates of volume and curvature. In

Section 4, the main results about Dehn surgery are deduced from the work in Sections 2 and 3. In Section 5, we obtain restrictions on the genus of surfaces in the complement of a hyperbolic knot in terms of their boundary slopes. In Section 6, we examine 3-manifolds which are 'almost hyperbolic', in the sense that for any $\delta > 0$, they have a complete finite volume Riemannian metric with all sectional curvatures between $-1-\delta$ and $-1+\delta$. We show that such 3-manifolds must have a complete finite volume hyperbolic structure. The proof of this result uses Dehn surgery in a crucial way.

2. When is a slope long?

Since the majority of theorems in this paper are stated in terms of slope length, we will now establish some conditions which imply that a slope is long. Throughout this section, we will examine slopes lying on a single torus T in ∂X. There may be boundary components of X other than T, but nevertheless the length of a slope on T is a well-defined topological invariant.

Recall that the *distance* $\Delta(s_1, s_2)$ between two slopes s_1 and s_2 on a torus is defined to be the minimum number of intersection points of two representative simple closed curves. The following lemma implies that if the distance between two slopes is large, then at least one of them must be long.

Lemma 2.1. *Let X be a hyperbolic 3-manifold with a torus T in its boundary. Let s_1 and s_2 be slopes on T. Then*

$$l(s_1)\, l(s_2) \geq \sqrt{3}\, \Delta(s_1, s_2).$$

Moreover, if all slopes on T have length at least L, say, then

$$l(s_1)\, l(s_2) \geq \sqrt{3}\, L^2 \Delta(s_1, s_2).$$

Proof. It is a well-known observation of Thurston that the length of each slope on T is at least 1. (See [4, Theorem 11] for example.) Hence, the first inequality of the lemma follows from the second. We pick a generating set for $H_1(T)$ as follows. We first assign arbitrary orientations to s_1 and s_2. We let $[s_1] \in H_1(T)$ be one generator, and extend this to a generating set by picking one further element $[s_3]$. Then,

$$[s_2] = \pm\Delta(s_1, s_2)[s_3] + n[s_1],$$

for some integer n. Let N be the maximal horoball neighbourhood of the cusp at T. Let $\mathbb{R}^2$ be the boundary in $\mathbb{H}^3$ of an associated horoball. Let P (respectively, P') be a fundamental domain in $\mathbb{R}^2$ for the group of covering translations generated by $[s_1]$ and $[s_2]$ (respectively, by $[s_1]$ and $[s_3]$). Note that

$$l(s_1)\, l(s_2) \geq \text{Area}(P) = \Delta(s_1, s_2)\text{Area}(P').$$

This formula is clear from Figure 1.

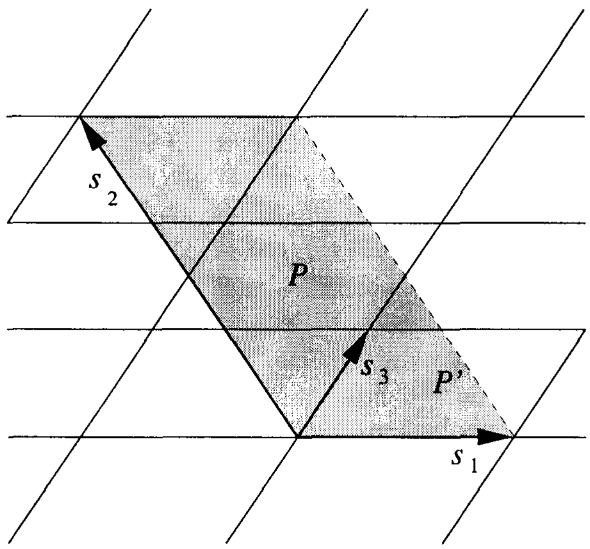

FIGURE 1

It is well known that the area of P' is at least $\sqrt{3}$ (see [1, Theorem 2]). However, the argument there readily implies that

$$\text{Area}(P') \geq \sqrt{3}\, L^2.$$

Hence, we deduce that

$$l(s_1)\, l(s_2) \geq \sqrt{3}\, L^2 \Delta(s_1, s_2).$$

q.e.d.

Now, there are various topological circumstances when a slope e is known to be 'short'. These are summarised in the following proposition.

Proposition 2.2. *Let X be a compact orientable hyperbolic 3-manifold and let T be a toral boundary component of X. Then a slope e on T has length no more than 2π if either of the following hold:*

1. $\mathrm{int}(X(e))$ *does not admit a complete finite volume negatively curved Riemannian metric (for example, $X(e)$ may be reducible, toroidal or Seifert fibred), or*

2. *the core of the filled-in solid torus in $X(e)$ has finite order in $\pi_1(X(e))$.*

Proof. Part (1) above is a mere restatement of the '2π' theorem, with the added assertion that if the interior of a compact orientable 3-manifold M admits a complete finite volume negatively curved metric, then M cannot be reducible, toroidal or Seifert fibred. This is well-known, but we sketch a proof. By the Hadamard-Cartan theorem [2], the universal cover of $\mathrm{int}(M)$ is homeomorphic to $\mathbb{R}^3$, and so M is irreducible. By [2], any $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\pi_1(M)$ is peripheral. Hence, M is atoroidal. Since $\mathrm{int}(M)$ is covered by $\mathbb{R}^3$, $\pi_1(M)$ is infinite. The fundamental group of any orientable Seifert fibre space either has a cyclic normal subgroup or is trivial [16]. However, the fundamental group of a complete negatively curved finite volume Riemannian manifold cannot have a cyclic normal subgroup [2]. Hence, M is not Seifert fibred.

To prove Part (2), we recall from the proof of the '2π' theorem [4] that if $l(e) > 2\pi$, then $\mathrm{int}(X(e))$ has a complete finite volume negatively curved metric, in which the core of the filled-in solid torus is a geodesic. But, in such a manifold, closed geodesics have infinite order in the fundamental group [2]. q.e.d.

We therefore make the following definition.

Definition 2.3. Let X be a hyperbolic 3-manifold and let e be a slope on a toral boundary component T of X. Then e is *short* if $l(e) \leq 2\pi$. We say that e is *minimal* if $l(s) \geq l(e)$ for all slopes s on T.

Note that there is always at least one minimal slope on T, but that it need not be short. The importance of the above definition is that if some slope s has large intersection number with a slope e which is either short or minimal, then we can deduce that the length of s is large. Moreover the bounds we construct are independent of the manifold X.

Corollary 2.4. *Let X be a compact hyperbolic 3-manifold, and let s be a slope on a torus component T of ∂X. If e is a short slope on T,* *then*

$$l(s) \geq \sqrt{3}\,\Delta(s,e)/2\pi.$$

If e is a minimal slope on T, then

$$l(s) \geq \sqrt{3}\,\Delta(s,e).$$

Proof. The first inequality is an immediate consequence of Lemma 2.1 and the definition of 'short'. If e is a minimal slope on T, then all slopes on T have length at least $l(e)$, and hence by Lemma 2.1,

$$l(s)\,l(e) \geq \sqrt{3}\,[l(e)]^2\Delta(s,e).$$

Thus, we get that

$$l(s) \geq \sqrt{3}\,l(e)\Delta(s,e) \geq \sqrt{3}\,\Delta(s,e),$$

since $l(e) \geq 1$ by [4, Theorem 11]. q.e.d.

3. Estimates of curvature, volume and Gromov norm

In this section, we compare the Gromov norm [17, Chapter 6] of a hyperbolic 3-manifold X with the Gromov norm of a 3-manifold obtained by Dehn filling tori $T_1, \ldots, T_n$ in ∂X.

In [17, Section 6.5], Thurston gave various definitions of the Gromov norm of a compact orientable 3-manifold X, with ∂X a (possibly empty) union of tori. We shall use the terminology $|X|$ for the quantity which Thurston calls $||[X, \partial X]||_0$. This is defined as follows. Consider the fundamental class $[X, \partial X]$ in the singular homology group $H_3(X, \partial X; \mathbb{R})$. If $z = \sum a_i\sigma_i$ is a representative of $[X, \partial X]$, where $a_i \in \mathbb{R}$ and each σ_i is a singular 3-simplex, then we consider the real number $||z|| = \sum |a_i|$. In the case where $\partial X = \emptyset$, the Gromov norm is defined to be

$$|X| = \inf\{||z|| : z \text{ represents } [X, \partial X]\}.$$

In the case where $\partial X \neq \emptyset$, a representative z of $[X, \partial X]$ determines a representative ∂z of $[\partial X] \in H_2(\partial X; \mathbb{R})$. Thurston defines

$$|X| = \lim_{a\to 0} \inf\{||z|| : z \text{ represents } [X, \partial X] \text{ and } ||\partial z|| \leq |a|\}$$

and shows that this limit exists. Crucially, the Gromov norm of X is a topological invariant.

We compare the Gromov norms of X and $X(s_1, \dots, s_n)$ (where $s_1, \dots, s_n$ are slopes on ∂X) by returning to the proof of the '2π' theorem. The idea behind this proof is simple. One first removes from X the interior of an almost maximal horoball neighbourhood N of the cusps at $T_1 \cup \cdots \cup T_n$. Then one glues back in solid tori V_i which have negatively curved Riemannian metrics agreeing near ∂V_i with that near ∂N. The following proposition deals with the sectional curvatures and the volume of the metric on each solid torus.

If M is a manifold with interior having a Riemannian metric k, let $\mathrm{Vol}(M, k)$ denote its volume and let $\kappa_{\inf}(M, k)$ (respectively, $\kappa_{\sup}(M, k)$) denote the infimum (respectively, the supremum) of its sectional curvatures.

Proposition 3.1. *For any two real numbers $\ell_1 > 2\pi$ and $\ell_2 > 0$, we may construct a Riemannian metric k on the solid torus V, with the following properties. In a collar neighbourhood of ∂V, the metric is hyperbolic. The boundary ∂V inherits a Euclidean metric $k|_{\partial V}$. The length in this metric of a shortest meridian curve C on ∂V is ℓ_1. The length of a (Euclidean) geodesic running perpendicularly from C to C is ℓ_2. Also,* $\mathrm{Vol}(V, k)/\mathrm{Vol}(\partial V, k|_{\partial V})$, $\kappa_{\inf}(V, k)$ *and* $\kappa_{\sup}(V, k)$ *are all independent of ℓ_2. But, there is a non-decreasing function $\alpha\colon (2\pi, \infty) \to (0, 1)$ such that*

$$-(\alpha(\ell_1))^{-1} \leq \kappa_{\inf}(V, k) < \kappa_{\sup}(V, k) \leq -\alpha(\ell_1),$$

$$\frac{\mathrm{Vol}(V, k)}{\mathrm{Vol}(\partial V, k|_{\partial V})/2} \geq \alpha(\ell_1).$$

Proof. In Bleiler and Hodgson's proof of the '2π' theorem [4], a Riemannian metric k is constructed on V which has most of these properties. They assign cylindrical co-ordinates $(r,\ \mu,\ \lambda)$ to V, where $r \leq 0$ is the radial distance measured *outward* from ∂V, $0 \leq \mu \leq 1$ is measured in the meridional direction and $0 \leq \lambda \leq 1$ is measured in a direction perpendicular to μ and r. The distance from the core of V to the boundary is $-r_0$, for some negative constant r_0. The Riemannian metric is denoted

$$ds^2 = dr^2 + [f(r)]^2 d\mu^2 + [g(r)]^2 d\lambda^2,$$

where $f\colon [r_0, 0] \to \mathbb{R}$ and $g\colon [r_0, 0] \to \mathbb{R}$ are functions. Graphs of f and g are given in Bleiler and Hodgson's paper [4], and are reproduced in Figure 2.

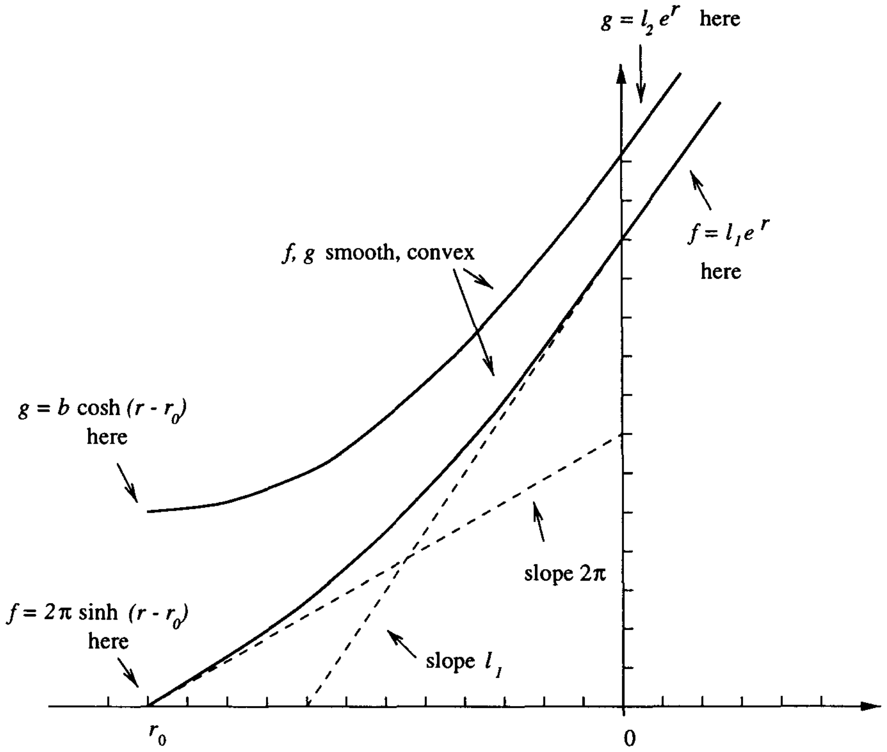

FIGURE 2

Making the substitutions $x_1 = r$, $x_2 = \mu$ and $x_3 = \lambda$, they calculate the sectional curvatures as

$$\kappa_{12} = -\frac{f''}{f},$$
$$\kappa_{13} = -\frac{g''}{g},$$
$$\kappa_{23} = -\frac{f' \cdot g'}{f \cdot g}.$$

They observe that, since f, f', f'', g, g' and g'' are all positive in the range $r_0 < r \leq 0$, the sectional curvatures are all negative. To ensure that the cone angle at the core of V is 2π, it is enough to ensure that the gradient of f at $r = r_0$ is 2π. Also, near $r = 0$, f and g are both exponential, which guarantees that the sectional curvatures are all -1 near ∂V.

Bleiler and Hodgson argue that, providing $\ell_1 > 2\pi$ and $\ell_2 > 0$, we may find a metric k satisfying the above properties. We can therefore pick α (which is a real-valued function of ℓ_1 and ℓ_2) as follows. Given ℓ_1 and ℓ_2, there is a Riemannian metric k on V satisfying all of the above properties, and a real number a (with $0 < a < 1$) for which the following inequalities hold:

$$-a^{-1} < \kappa_{\text{inf}}(V,k) < \kappa_{\text{sup}}(V,k) < -a,$$

$$\frac{\text{Vol}(V,k)}{\text{Vol}(\partial V, k|_{\partial V})/2} > a.$$

We define $\alpha(\ell_1, \ell_2)$ to be *half* the supremum of a, where a satisfies the above inequalities, and k satisfies the above conditions. There is a good deal of freedom over the choice of α. We picked half the supremum simply because it is less than the supremum, and hence there is some metric k for which

$$-(\alpha(\ell_1,\ell_2))^{-1} < \kappa_{\text{inf}}(V,k) < \kappa_{\text{sup}}(V,k) < -\alpha(\ell_1,\ell_2),$$

$$\frac{\text{Vol}(V,k)}{\text{Vol}(\partial V, k|_{\partial V})/2} > \alpha(\ell_1,\ell_2).$$

At this stage, α depends on both ℓ_1 and ℓ_2. But we shall show that α is independent of ℓ_2 and is a non-decreasing function of ℓ_1.

Given a metric k on V

$$ds^2 = dr^2 + [f(r)]^2 d\mu^2 + [g(r)]^2 d\lambda^2,$$

we can define another metric

$$ds_0^2 = dr^2 + [f(r)]^2 d\mu^2 + c^2[g(r)]^2 d\lambda^2,$$

for any positive real constant c. Using the formulae for the sectional curvatures, we see that this alteration leaves the sectional curvatures unchanged. It does not alter the ratio of $\text{Vol}(V,k)$ and $\text{Vol}(\partial V, k|_{\partial V})$. It leaves ℓ_1 unchanged, but scales ℓ_2 by a factor of c. Hence, α is independent of ℓ_2, and we therefore refer to α as a function of the single variable ℓ_1. Note that we could not have made a similar argument with ℓ_1, since it is vital that the cone angle at the core of V is 2π.

It remains to show that α is a non-decreasing function of ℓ_1. Suppose V has a metric k as above. Then we can enlarge V (creating a bigger solid torus V' with metric k') by letting r vary in the range $r_0 \leq r \leq c$, for any positive real constant c, and defining $f(r) = \ell_1 e^r$ and $g(r) = \ell_2 e^r$ for $r \geq 0$. In other words, we attach a collar to ∂V, with the metric being hyperbolic in the collar. The length of the shortest meridian curve on the boundary of the new solid torus V' is $\ell_1 e^c$. If the metric k on V satisfies

$$-a^{-1} < \kappa_{\inf}(V,k) < \kappa_{\sup}(V,k) < -a,$$

$$\frac{\mathrm{Vol}(V,k)}{\mathrm{Vol}(\partial V, k|_{\partial V})/2} > a,$$

(with $0 < a < 1$), then the metric on the enlarged solid torus satisfies the same inequalities. The inequality regarding volumes requires some explanation:

$$\begin{aligned}\mathrm{Vol}(V',k') &= \mathrm{Vol}(V,k) + \int_0^c f(r)g(r)dr \\ &= \mathrm{Vol}(V,k) + \ell_1\ell_2(e^{2c}-1)/2 \\ &> a\mathrm{Vol}(\partial V, k|_{\partial V})/2 + \ell_1\ell_2(e^{2c}-1)/2 \\ &> a[\mathrm{Vol}(\partial V, k|_{\partial V}) + \ell_1\ell_2(e^{2c}-1)]/2 \\ &= a\mathrm{Vol}(\partial V', k'|_{\partial V'})/2.\end{aligned}$$

Therefore, for any $c > 0$, $\alpha(\ell_1 e^c) \geq \alpha(\ell_1)$, and hence α is a non-decreasing function. q.e.d.

It is actually possible to define a function α satisfying the conditions of Proposition 3.1 for which $\alpha(\ell_1) \to 1$ as $\ell_1 \to \infty$. In other words, if ℓ_1 is sufficiently large, then we can construct a metric k on V for which the sectional curvatures approach -1 and the volume approaches that of a cusp. This is intuitively plausible from the graphs of f and g. However, a rigorous proof of this result is slightly technical and long-winded. Since we will not actually need this result, we offer only a brief summary of the proof.

The idea is to construct, for any t with $0 < t < 1$, the functions f and g in terms of a certain differential equation, which we omit here. This differential equation in fact guarantees that

$$-1 - t \leq \kappa_{\inf}(V,k) \leq \kappa_{\sup}(V,k) \leq -1 + t.$$

The constant r_0 is defined to be the value of r for which $f(r) = 0$. The condition that $f'(r_0) = 2\pi$ determines $f(0)$, which is ℓ_1. Hence, we obtain ℓ_1 as a function of t. One shows that ℓ_1 lies in the range $(2\pi, \infty)$, and that there exists an inverse function $t\colon (2\pi,\infty) \to (0,1)$. One also shows that as ℓ_1 tends to ∞, the associated t tends to zero. In addition, the definition of the metric k ensures that as $\ell_1 \to \infty$, the ratio

$$\frac{\mathrm{Vol}(V,k)}{\mathrm{Vol}(\partial V, k|_{\partial V})/2}$$

tends to 1. Hence, it is straightforward to construct the function α satisfying the conditions of Proposition 3.1 and also $\alpha(\ell_1) \to 1$ as $\ell_1 \to \infty$.

We now apply Proposition 3.1 to $X(s_1, \dots, s_n)$, where X is a hyperbolic 3-manifold and $s_1, \dots, s_n$ are slopes on distinct components of ∂X. The following proposition analyses the volume and sectional curvatures of $X(s_1, \dots, s_n)$.

Proposition 3.2. *Let $\alpha\colon (2\pi, \infty) \to (0,1)$ be the function in Proposition 3.1. Let X be a compact 3-manifold with interior having a complete finite volume hyperbolic metric h, and let $s_1, \dots, s_n$ be slopes on distinct tori $T_1, \dots, T_n$ in ∂X. Suppose that there is a maximal horoball neighbourhood of $T_1 \cup \cdots \cup T_n$ on which $l(s_i) > 2\pi$ for each i. Let $\ell = \min_{1 \le i \le n} l(s_i)$. Then $X(s_1, \dots, s_n)$ has a complete finite volume negatively curved Riemannian metric g for which the following formulae hold:*

$$-(\alpha(\ell))^{-1} \le \kappa_{\inf}(X(s_1, \dots, s_n), g),$$
$$\kappa_{\sup}(X(s_1, \dots, s_n), g) \le -\alpha(\ell),$$
$$\alpha(\ell) < \frac{\mathrm{Vol}(X(s_1, \dots, s_n), g)}{\mathrm{Vol}(X, h)}.$$

Proof. We follow the proof of the '2π' theorem. Let $\bigcup_{i=1}^n B_i$ be a union of horoballs in $\mathbb{H}^3$, which projects to a maximal horoball neighbourhood N of $T_1 \cup \cdots \cup T_n$. Suppose that B_i projects to the cusp at T_i. Let T_i' be the quotient of ∂B_i by the subgroup of parabolic isometries in $\pi_1(X)$ which preserve B_i. Now construct as in Proposition 3.1 a metric k_i on the solid torus V_i which agrees on ∂V_i with the Euclidean metric on T_i', and which has meridian length $l(s_i)$. The Riemannian metric g on $X(s_1, \dots, s_n)$ is just that obtained by attaching $\bigcup_{i=1}^n (V_i, k_i)$ to $(\mathrm{int}(X) - \mathrm{int}(N), h)$.

The first formula of Proposition 3.1 immediately implies the first two formulae of the proposition. To obtain the third formula, first note that it is an elementary calculation that

$$\mathrm{Vol}(N, h) = \sum_{i=1}^n \mathrm{Vol}(T_i', h|_{T_i'})/2.$$

Also, the metrics on T_i' and ∂V_i agree:

$$\mathrm{Vol}(T_i', h|_{T_i'}) = \mathrm{Vol}(\partial V_i, k_i|_{\partial V_i}).$$

Proposition 3.1 gives that

$$\begin{aligned}\mathrm{Vol}(V_i, k_i) &\geq \alpha(l(s_i))\mathrm{Vol}(\partial V_i, k_i|_{\partial V_i})/2 \\ &\geq \alpha(\ell)\mathrm{Vol}(\partial V_i, k_i|_{\partial V_i})/2,\end{aligned}$$

since α is a non-decreasing function. Hence,

$$\sum_{i=1}^{n} \mathrm{Vol}(V_i, k_i) \geq \alpha(\ell) \sum_{i=1}^{n} \mathrm{Vol}(\partial V_i, k_i|_{\partial V_i})/2.$$

So,

$$\begin{aligned}\mathrm{Vol}(X(s_1, \dots, s_n), g) &= \mathrm{Vol}(X - N, h) + \sum_{i=1}^{n} \mathrm{Vol}(V_i, k_i) \\ &\geq \mathrm{Vol}(X - N, h) + \alpha(\ell) \sum_{i=1}^{n} \mathrm{Vol}(\partial V_i, k_i|_{\partial V_i})/2 \\ &> \alpha(\ell)[\mathrm{Vol}(X - N, h) + \sum_{i=1}^{n} \mathrm{Vol}(\partial V_i, k_i|_{\partial V_i})/2] \\ &= \alpha(\ell)[\mathrm{Vol}(X - N, h) + \mathrm{Vol}(N, h)] \\ &= \alpha(\ell)[\mathrm{Vol}(X, h)],\end{aligned}$$

which establishes the final formula of the proposition. q.e.d.

The aim now is to use the comparison of volumes and sectional curvatures of (X, h) and $(X(s_1, \dots, s_n), g)$ to make a comparison of their Gromov norms.

Proposition 3.3. *There is a non-increasing function $\beta\colon (2\pi, \infty) \to (1, \infty)$, which has the following property. Let X be a compact hyperbolic 3-manifold and let $s_1, \dots, s_n$ be slopes on distinct components $T_1, \dots, T_n$ of ∂X. Suppose that there is a maximal horoball neighbourhood of $T_1 \cup \dots \cup T_n$ on which $l(s_i) > 2\pi$ for each i. Then*

$$|X(s_1, \dots, s_n)| \leq |X| < |X(s_1, \dots, s_n)|\, \beta(\min_{1 \leq i \leq n} l(s_i)).$$

Proof. It is a corollary of [17, 6.5.2] that $|X(s_1,\ldots,s_n)| \leq |X|$. To prove the second half of the inequality of the proposition, we need to compare the volume of a negatively curved Riemannian manifold with its Gromov norm. For a manifold X with interior having a complete finite volume hyperbolic metric h, it is proved in [17, 6.5.4] that

$$\mathrm{Vol}(X,h) = v_3\,|X|,$$

where v_3 is the volume of a regular ideal 3-simplex in $\mathbb{H}^3$. Here, by the '2π' theorem $X(s_1,\ldots,s_n)$ can be given a negatively curved metric g, and so we need a version of Gromov's result which applies in this case. It is possible to show [3, C.5.8] that if M is a 3-manifold with a complete finite volume Riemannian metric g which has $\kappa_{\sup}(M,g) \leq -1$, then

$$|M|\,\pi/2 \geq \mathrm{Vol}(M,g).$$

Now, if the metric g on $X(s_1,\ldots,s_n)$ is scaled by a positive constant λ to give a metric λg, then it is an elementary consequence of the definition of sectional curvature that

$$\kappa_{\sup}(X(s_1,\ldots,s_n),\lambda g) = \lambda^{-2}\kappa_{\sup}(X(s_1,\ldots,s_n),g).$$

Hence, by letting $\lambda = \sqrt{-\kappa_{\sup}(X(s_1,\ldots,s_n),g)}$, we obtain that

$$\begin{aligned}|X(s_1,\ldots,s_n)|\pi/2 &\geq \mathrm{Vol}(X(s_1,\ldots,s_n),\lambda g)\\ &= \lambda^3\mathrm{Vol}(X(s_1,\ldots,s_n),g)\\ &= (-\kappa_{\sup}(X(s_1,\ldots,s_n),g))^{3/2}\mathrm{Vol}(X(s_1,\ldots,s_n),g).\end{aligned}$$

We can now use Proposition 3.2 to deduce that

$$\begin{aligned}&\kappa_{\sup}(X(s_1,\ldots,s_n),g) \leq -\alpha(\ell),\\ &\mathrm{Vol}(X(s_1,\ldots,s_n),g) > \alpha(\ell)[\mathrm{Vol}(X,h)],\end{aligned}$$

where α is the function given in Proposition 3.1 and $\ell = \min_{1\leq i\leq n} l(s_i)$. So,

$$\begin{aligned}|X(s_1,\ldots,s_n)|\pi/2 &> [\alpha(\ell)]^{5/2}\mathrm{Vol}(X,h)\\ &= [\alpha(\ell)]^{5/2}v_3\,|X|.\end{aligned}$$

The proposition is now proved by letting $\beta(x) = [\alpha(x)]^{-5/2}\pi/2v_3$. q.e.d.

It should be possible to find a function β satisfying the requirements of Proposition 3.3 and for which $\beta(\ell_1) \to 1$ as $\ell_1 \to \infty$. To find such a β, one examines the volume of a straight 3-simplex Δ in a simply-connected 3-manifold with sectional curvatures between $-1-\delta$ and $-1+\delta$ for sufficiently small $\delta > 0$. If one shows that as $\delta \to 0$, the maximal volume of Δ tends to v_3, then one can find a β satisfying the conditions of Proposition 3.3, for which $\beta(\ell_1) \to 1$ as $\ell_1 \to \infty$.

4. Applications to Dehn surgery

We now use the estimates of the previous section to deduce some new results about Dehn surgery. The most far-reaching of these is the following theorem.

Theorem 4.1. *Let M be a compact orientable 3-manifold, with ∂M a (possibly empty) union of tori. Let X be a hyperbolic manifold and let $s_1, \dots, s_n$ be a collection of slopes on n distinct tori $T_1, \dots, T_n$ in ∂X, such that $X(s_1, \dots, s_n)$ is homeomorphic to M. Suppose that there exists in $\mathrm{int}(X)$ a maximal horoball neighbourhood of $T_1 \cup \dots \cup T_n$ on which each slope s_i has length at least $2\pi + \epsilon$, for some $\epsilon > 0$. Then, for any given M and ϵ, there is only a finite number of possibilities (up to isometry) for X, n and $s_1, \dots, s_n$.*

To prove this, we shall need two well-known lemmas.

Lemma 4.2. *Let X be a compact orientable hyperbolic 3-manifold, and let $T_1, \dots, T_n$ be a collection of tori in ∂X. For each $i \in \mathbb{N}$ and $j \in \{1, \dots, n\}$, let s_i^j be a slope on T_j. Assume that, for every j, $s_i^j \neq s_k^j$ if $i \neq k$. Then any given 3-manifold M is homeomorphic to $X(s_i^1, \dots, s_i^n)$ for at most finitely many i.*

Proof. By the hyperbolic Dehn surgery theorem of Thurston [3, Theorem E.5.1], $X(s_i^1, \dots, s_i^n)$ is hyperbolic for i sufficiently large. Thus, if the theorem were not true, then M would have to be hyperbolic. Moreover, from the proof of Thurston's hyperbolic Dehn surgery theorem, for i sufficiently large, the cores of the filled-in solid tori in $X(s_i^1, \dots, s_i^n)$ are geodesics, whose lengths each tend to zero, as $i \to \infty$. In particular, for i sufficiently large, $X(s_i^1, \dots, s_i^n)$ has a geodesic shorter than the shortest geodesic in M. This is impossible by Mostow rigidity [3, Theorem C.5.4]. q.e.d.

Lemma 4.3. *Let X be a compact orientable hyperbolic 3-manifold, and let $T_1, \dots, T_n$ be a collection of tori in ∂X. Let*

$$\{(s_i^1, \dots, s_i^n) : i \in \mathbb{N}\}$$

be a sequence of distinct n-tuples, where each s_i^j is a slope on T_j. Suppose that, for each i, we can find a maximal horoball neighbourhood of the cusps at $T_1 \cup \cdots \cup T_n$ on which each s_i^j has length more than 2π. Then any given 3-manifold M is homeomorphic to $X(s_i^1, \dots, s_i^n)$ for at most finitely many i.

Proof. If the lemma were not true, we could pass to a subsequence, such that M is homeomorphic to $X(s_i^1, \dots, s_i^n)$ for each i. There are two possibilities for each $j \in \{1, \dots, n\}$:

(i) the sequence $\{s_i^j : i \in \mathbb{N}\}$ contains a subsequence in which the slopes s_i^j are all distinct, or

(ii) the sequence $\{s_i^j : i \in \mathbb{N}\}$ runs through only finitely many slopes.

Since the n-tuples $(s_i^1, \dots, s_i^n)$ are distinct, at least one j satisfies (i). After re-ordering, we may assume this value of j is n. Pass to this subsequence, where the slopes s_i^n are all distinct. In this new sequence, the integers $j \in \{1, \dots, n-1\}$ either satisfy (i) or (ii). If some j satisfies (i), say $j = n-1$, pass to this subsequence. Continuing in this fashion, we obtain a sequence and an integer $m \geq 1$ such that

(i) $s_i^j \neq s_k^j$ for $i \neq k$ and $j \geq m$, and

(ii) $\{s_i^j : i \in \mathbb{N}\}$ runs through only finitely many slopes, for each $j < m$.

By passing to a subsequence, we may assume that s_i^j is the same slope s^j for all i, when $j < m$. If $m > 1$, let $Y = X(s^1, \dots, s^{m-1})$. Otherwise, let $Y = X$. Now, we may find a maximal horoball neighbourhood of the cusps at $T_1 \cup \cdots \cup T_n$ on which each s^j has length more than 2π. Hence, Y admits a negatively curved metric, by the '2π' theorem. Hence, it cannot be reducible, toroidal or Seifert fibred. (See the proof of Proposition 2.2.) Its boundary contains $T_m \cup \cdots \cup T_n$ and so is non-empty. Hence, by Thurston's theorem on the geometrisation of Haken 3-manifolds [12, Chapter V], Y is hyperbolic. But, M is homeomorphic to $Y(s_i^m, \dots, s_i^n)$ for each i. Lemma 4.2 gives us a contradiction. q.e.d.

Proof of Theorem 4.1. Suppose that, on the contrary, there exists a sequence of 3-manifolds X_i with complete finite volume hyperbolic metrics h_i on their interiors, and slopes $s_i^1, \dots, s_i^{n(i)}$ on ∂X_i with

$l(s_i^j) \geq 2\pi + \epsilon$, such that $X_i(s_i^1, \ldots, s_i^{n(i)})$ is homeomorphic to M. Then, by Proposition 3.3,

$$|X_i| < |M|\,\beta(\min_{1\leq j\leq n(i)} l(s_i^j)).$$

Since β is a non-increasing function,

$$|X_i| < |M|\,\beta(2\pi + \epsilon).$$

Thus, the sequence $|X_i|$ is bounded, and so the sequence $\mathrm{Vol}(X_i, h_i)$ is also bounded, since the Gromov norm and the volume of a hyperbolic 3-manifold are proportional [17, 6.5.4]. But for any real number c, the collection of complete orientable hyperbolic 3-manifolds with volume at most c is a compact topological space when endowed with the geometric topology [3, Theorem E.1.10]. Hence, we may pass to a subsequence (also denoted $\{X_i\}$), such that $\mathrm{int}(X_i)$ converges in the geometric topology to a complete finite volume hyperbolic 3-manifold $\mathrm{int}(X_\infty)$, say, where X_∞ is compact and orientable. This implies (see [3, Theorem E.2.4]) that, for i sufficiently large, the following is true. In each 3-manifold $\mathrm{int}(X_i)$, there is a (possibly empty) union L_i of disjoint closed geodesics, such that $\mathrm{int}(X_i) - L_i$ is diffeomorphic to $\mathrm{int}(X_\infty)$. This diffeomorphism is a k_i-bi-Lipschitz map except in a small neighbourhood of L_i, for real numbers $k_i \geq 1$ which tend to 1, as $i \to \infty$. This diffeomorphism also takes a maximal horoball neighbourhood N_i of cusps of $\mathrm{int}(X_i)$ to a neighbourhood of cusps of $\mathrm{int}(X_\infty)$ which closely approximates a maximal horoball neighbourhood N_i'. We extend N_i' to a maximal horoball neighbourhood N_i'' of all the cusps of $\mathrm{int}(X_\infty)$. The slopes $s_i^1, \ldots, s_i^{n(i)}$ correspond to slopes $\sigma_i^1, \ldots, \sigma_i^{n(i)}$ on toral components of ∂X_∞, and the ratios $l(s_i^j)/l(\sigma_i^j) \to 1$ as $i \to \infty$, where the lengths $l(s_i^j)$ and $l(\sigma_i^j)$ are measured on N_i and N_i'' respectively. Hence, as $l(s_i^j) \geq 2\pi + \epsilon$, $l(\sigma_i^j) > 2\pi$ for i sufficiently large. Since X_∞ has a finite number of cusps, the sequences $n(i)$ and $|L_i|$ are bounded. Hence, by passing to a subsequence, we may assume that $n(i)$ is some fixed positive integer n and that $|L_i|$ is some fixed non-negative integer p, for all i. We may pass to a further subsequence where for any j, σ_i^j lies on the same torus for all i. Now, $p > 0$, for otherwise, $X_\infty = X_i$ for all i and then M is homeomorphic to $X_\infty(\sigma_i^1, \ldots, \sigma_i^n)$ for each i. This is a contradiction by Lemma 4.3. Thus there are slopes $(t_i^1, \ldots, t_i^p)$ on ∂X_∞ such that $X_\infty(t_i^1, \ldots, t_i^p)$ is homeomorphic to X_i. We may assume that, for any j, the slopes t_i^j lie on the same torus P^j for all i. Since the manifolds

$\mathrm{int}(X_i)$ converge in the geometric topology to $\mathrm{int}(X_\infty)$, we may assume that the slopes t_i^j are all distinct. We shall now show that, for each j, $l(t_i^j) \to \infty$, as $i \to \infty$, where the slope lengths are measured on N_i''. For each torus T^k in ∂X_∞, let N^k be the maximal horoball neighbourhood of T^k. Then, we may find a horoball neighbourhood H^j of P^j which misses all N^k other than the horoball N^j corresponding to P^j. Thus, H^j lies inside N_i'' for each i. Now, the lengths of t_i^j tend to ∞ as $i \to \infty$, where the length is measured on H^j, since the slopes t_i^j are all distinct. Hence, the lengths $l(t_i^j) \to \infty$, as $i \to \infty$, where the slope lengths are measured on N_i''. So, M is homeomorphic to $X_\infty(t_i^1, \dots, t_i^p, \sigma_i^1, \dots, \sigma_i^n)$ for each i, and for sufficiently large i, $l(t_i^j) > 2\pi$ and $l(\sigma_i^j) > 2\pi$ for each j, where the slope lengths are measured on N_i''. This is a contradiction, by Lemma 4.3. q.e.d.

Theorem 4.1 has the following corollary.

Theorem 4.4. *Let M be a compact orientable 3-manifold with ∂M a (possibly empty) union of tori. Suppose that M is homeomorphic to $X(s)$, where X is a hyperbolic 3-manifold and s is a slope on a toral boundary component T of X. Suppose also that e is a short slope on T such that $\Delta(s,e) > 22$, or that e is a minimal slope with $\Delta(s,e) > 3$. Then, for a given M, there is only a finite number of possibilities (up to isometry) for X, s and e.*

Proof. Suppose that e is a short slope with $\Delta(s,e) > 22$. The proof when e is minimal is entirely analagous. Fix ϵ as $(23\sqrt{3}/2\pi) - 2\pi$, which is greater than zero. By Theorem 4.1, there is only a finite number of hyperbolic 3-manifolds X and slopes s on a torus component of ∂X, such that $X(s)$ is homeomorphic to M and such that $l(s) \geq 2\pi + \epsilon$. But if e is a short slope on T with $\Delta(s,e) \geq 23$, then, by Corollary 2.4,

$$l(s) \geq \frac{\sqrt{3}\,\Delta(s,e)}{2\pi} \geq \frac{23\sqrt{3}}{2\pi} = 2\pi + \epsilon.$$

Thus, there is only a finite number of possibilities for X and s. Also there is only a finite number of short slopes e on T. Hence, the theorem is proved. q.e.d.

Corollary 4.5. *For a given closed orientable 3-manifold M, there is at most a finite number of hyperbolic knots K in S^3 and fractions p/q (in their lowest terms) such that M is obtained by p/q-Dehn surgery along K and $|q| > 22$.*

Proof. The meridian slope e on $\partial\mathcal{N}(K)$ is a short slope, and $\Delta(e, p/q) = |q|$. Now apply Theorem 4.4. q.e.d.

Theorem 4.6. *Let M_1 and M_2 be compact orientable 3-manifolds with ∂M_i a (possibly empty) union of tori. Let X be a hyperbolic 3-manifold and let T be a toral boundary component of X. Suppose that there are slopes s_1 and s_2 on T, with $\Delta(s_1, s_2) > 22$, such that $X(s_i)$ is homeomorphic to M_i for $i = 1$ and 2. Then, for any given M_1 and M_2, there is only a finite number of possibilities (up to isometry) for X, s_1 and s_2.*

Proof. Suppose that there is an infinite number of triples (X, s_1, s_2), for which $X(s_i)$ is homeomorphic to M_i (for $i = 1$ and 2) and which have $\Delta(s_1, s_2) \geq 23$. Let $\epsilon = (\sqrt[4]{3}\sqrt{23}) - 2\pi$. If $l(s_1) < 2\pi + \epsilon$ and $l(s_2) < 2\pi + \epsilon$, then

$$l(s_1)\, l(s_2) < (2\pi + \epsilon)^2 = 23\sqrt{3} \leq \Delta(s_1, s_2)\sqrt{3},$$

but this cannot occur, by Lemma 2.1. Hence, an infinite number of the slopes s_1 or s_2 must have length at least $2\pi + \epsilon$. For the sake of definiteness, assume $l(s_1) \geq 2\pi + \epsilon$, for an infinite number of slopes s_1. By Theorem 4.1, there is only a finite number of possibilities for X and s_1. Hence, by passing to an infinite subcollection, we can find a fixed hyperbolic manifold X and an infinite number of distinct slopes s_2 such that $X(s_2)$ is homeomorphic to M_2. This contradicts Lemma 4.2. q.e.d.

The next result is a 'uniqueness' theorem for Dehn surgery. It should be compared with [11, Theorem 4.1].

Theorem 4.7. *For $i = 1$ and 2, let X_i be a hyperbolic 3-manifold and let s_i be a slope on ∂X_i. Then, there is a real number $C(X_1)$ depending only on X_1, such that if $l(s_2) > C(X_1)$, then*

$$\{X_1(s_1) \cong X_2(s_2)\} \iff \{(X_1, s_1) \cong (X_2, s_2)\},$$

where $\cong$ denotes a homeomorphism.

Proof. Suppose that there is no such real number $C(X_1)$. Then, there is a sequence of hyperbolic 3-manifolds X_2^i and slopes s_2^i on ∂X_2^i such that $l(s_2^i) \to \infty$, together with slopes s_1^i on ∂X_1, with the property that $X_1(s_1^i) \cong X_2^i(s_2^i)$, but $(X_1, s_1^i) \not\cong (X_2^i, s_2^i)$.

Note first that the sequence s_1^i can have no constant subsequence. For, if there were such a subsequence, say with slope s_1, then $X_1(s_1) \cong X_2^i(s_2^i)$ for infinitely many i. This contradicts Theorem 4.1.

Case 1. X_2^i runs through only finitely many hyperbolic 3-manifolds (up to isometry).

Then, by passing to a subsequence, we may assume that X_2^i is some fixed hyperbolic manifold X_2. Since s_1^i cannot run through only finitely many slopes, we may pass to a subsequence where $s_1^i \neq s_1^j$ if $i \neq j$. By Thurston's hyperbolic Dehn surgery theorem, for i sufficiently large, $X_1(s_1^i)$ is hyperbolic, with the core of the surgery solid torus being the unique shortest geodesic. Similarly, for i sufficiently large, $X_2(s_2^i)$ is hyperbolic, with the core of the surgery solid torus being the unique shortest geodesic. But, then by Mostow Rigidity, there is a homeomorphism from $X_1(s_1^i)$ to $X_2(s_2^i)$ which takes one geodesic to the other. Hence, $(X_1, s_1^i) \cong (X_2, s_2^i)$, which is contrary to assumption. Hence, the following case must hold.

Case 2. There is a subsequence in which X_2^i and X_2^j are not isometric if $i \neq j$.

Now, by Proposition 3.3,

$$(\beta(l(s_2^i)))^{-1}\,|X_2^i| < |X_2^i(s_2^i)| = |X_1(s_1^i)| \leq |X_1|.$$

Since β is a non-increasing function, the sequence $|X_2^i|$ is bounded, and so, by passing to a subsequence, we may assume that the 3-manifolds $\mathrm{int}(X_2^i)$ converge in the geometric topology to a hyperbolic 3-manifold $\mathrm{int}(X_2^\infty)$, where X_2^∞ is compact and orientable. For i sufficiently large, there is a union L_i of $n > 0$ disjoint closed geodesics in $\mathrm{int}(X_2^i)$, such that $\mathrm{int}(X_2^i) - L_i$ is diffeomorphic to $\mathrm{int}(X_2^\infty)$. In the complement of a small neighbourhood of L_i, this map is k_i-Lipschitz for real numbers $k_i \geq 1$ which tend to 1. Let $(t_1^i, \ldots, t_n^i)$ be the slopes on ∂X_2^∞ such that $X_2^\infty(t_1^i, \ldots, t_n^i)$ is homeomorphic to X_2^i. By passing to a subsequence, we may ensure that the slopes t_j^i are all distinct. Now, the slopes s_2^i correspond to slopes σ_2^i, say on ∂X_2^∞. Since the lengths $l(s_2^i) \to \infty$, so also the lengths $l(\sigma_2^i) \to \infty$. Hence, by Thurston's hyperbolic Dehn surgery theorem, for any $\epsilon > 0$, $X_2^\infty(\sigma_2^i, t_1^i, \ldots, t_n^i)$ is hyperbolic and the cores of the $(n+1)$ filled-in solid tori are geodesics of length less than ϵ, if i is sufficiently large. However, $X_2^\infty(\sigma_2^i, t_1^i, \ldots, t_n^i)$ is homeomorphic to $X_1(s_1^i)$. Since s_1^i has no constant subsequence, Thurston's hyperbolic Dehn surgery theorem gives that there is an integer N and an $\epsilon > 0$ such that, for all $i \geq N$, $X_1(s_1^i)$ is hyperbolic and the core of the filled-in solid torus is the unique geodesic with length less than ϵ. This is a contradiction. q.e.d.

Theorem 4.1 also has the following corollary regarding branched covers.

Corollary 4.8. *Let M be a compact orientable 3-manifold with ∂M a (possibly empty) union of tori, which is obtained as a branched cover of a compact orientable 3-manifold Y over a hyperbolic link L, via a representation $\rho\colon \pi_1(Y - L) \to S_r$. Suppose that the branching index of every lift of every component of $\partial\mathcal{N}(L)$ is at least 7. Then, for a given M, there are only finitely many possibilities for Y, L, r and ρ.*

Proof. The representation $\rho\colon \pi_1(Y - L) \to S_r$ determines a cover X of $Y - \text{int}(\mathcal{N}(L))$, and M is obtained from X by Dehn filling along slopes $s_1, \dots, s_n$ in ∂X. The hyperbolic structure on $Y - \text{int}(\mathcal{N}(L))$ lifts to a hyperbolic structure on X, and a maximal horoball neighbourhood of $\partial\mathcal{N}(L)$ lifts to a horoball neighbourhood of cusps of X. Since the length of each slope on $\partial\mathcal{N}(L)$ in $Y - \text{int}(\mathcal{N}(L))$ is at least 1 [4], the length of each s_i on N is at least 7. Thus Theorem 4.1 implies that there are only finitely many possibilities for X and $s_1, \dots s_n$. Now,

$$\text{Vol}(X) = r\text{Vol}(Y - \text{int}(\mathcal{N}(L))),$$

where r is the index of the cover $X \to Y - \text{int}(\mathcal{N}(L))$. There is a lower bound on $\text{Vol}(Y - \text{int}(\mathcal{N}(L)))$, since the volume of a maximal horoball neighbourhood of $\partial\mathcal{N}(L)$ is at least $\sqrt{3}$ [1]. Hence, for a given X, there is an upper bound on r. Once r and X are fixed, so is $\text{Vol}(Y - \text{int}(\mathcal{N}(L)))$. There are only finitely many hyperbolic manifolds of a given volume [3], and so there are only finitely many possibilities for $Y - \text{int}(\mathcal{N}(L))$. The lengths of the meridian slopes on $\partial\mathcal{N}(L)$ are bounded above by the lengths of the slopes s_i on N. Hence, there are only finitely many possibilities for Y and L. A representation $\rho\colon \pi_1(Y - L) \to S_r$ is determined by the image of a generating set of $\pi_1(Y - L)$. Hence, once Y, L and r are fixed, there are only finitely many possibilities for ρ. q.e.d.

5. The length of boundary slopes

The results about Dehn surgery in Section 4 bear a strong resemblance to the work in [11]. In that paper, the main theorem (1.4) of [10] was crucial in establishing strong restrictions on the number of intersection points between embedded surfaces in a 3-manifold and certain surgery curves. In this section, we use hyperbolic techniques to prove similar results. The main theorem of this section is the following.

Theorem 5.1. *Let X be a hyperbolic 3-manifold and let T be a toral component of ∂X. Let $f\colon F \to X$ be a map of a compact connected surface F into X, such that $f(F) \cap \partial X = f(\partial F)$. Suppose that $f_*\colon \pi_1(F) \to \pi_1(X)$ is injective, and that every essential arc in F maps to an arc in X which cannot be homotoped (rel its endpoints) into ∂X. Suppose also that $f(F) \cap T$ is a non-empty collection of disjoint simple closed curves, each with slope s. Then*

$$l(s)\,|f(F) \cap T| < -2\pi\,\chi(F).$$

This result has a number of corollaries, which include the following.

Corollary 5.2. *Let p/q be the boundary slope of an incompressible boundary-incompressible non-planar orientable surface F properly embedded in the exterior of a hyperbolic knot in S^3. Then*

$$|q| < \frac{4\pi^2\,\mathrm{genus}(F)}{\sqrt{3}}.$$

Proof. By Theorem 5.1,

$$l(p/q)\,|\partial F| < -2\pi\,\chi(F) = 2\pi(2\,\mathrm{genus}(F) - 2 + |\partial F|),$$

and so

$$(l(p/q) - 2\pi)|\partial F| < 4\pi(\mathrm{genus}(F) - 1).$$

Now, the inequality of the corollary is automatically satisfied when $|q| = 1$, since $\mathrm{genus}(F) > 0$. Hence we may assume that $|\partial F| \geq 2$. Moreover, if $l(p/q) \geq 2\pi$, then

$$2l(p/q) - 4\pi \leq (l(p/q) - 2\pi)|\partial F| < 4\pi(\mathrm{genus}(F) - 1).$$

If $l(p/q) < 2\pi$, then

$$2l(p/q) - 4\pi < 0 \leq 4\pi(\mathrm{genus}(F) - 1).$$

So, in either case,

$$l(p/q) < 2\pi\,\mathrm{genus}(F).$$

By Proposition 2.2, the meridian slope on $\partial\mathcal{N}(K)$ is short. By Corollary 2.4,

$$l(p/q) \geq |q|\,\sqrt{3}/2\pi.$$

Hence, we obtain the inequality of the corollary. q.e.d.

Let F be as in Theorem 5.1. Then F is neither a disc, nor a Möbius band, nor an annulus. Since $[s] \in \pi_1(X)$ is non-trivial, F cannot be a disc. If F were a Möbius band or an annulus, then the map f could be homotoped (rel ∂F) to a map into ∂X, as X is hyperbolic. Since F contains an essential arc, this is contrary to assumption.

We may therefore pick an ideal triangulation of $\mathrm{int}(F)$. In other words, we may express $\mathrm{int}(F)$ as a union of 2-simplices glued along their edges, with the 0-simplices then removed. We may also ensure that each 1-cell in F is an essential arc. To see that such an ideal triangulation exists, fill in the boundary components of F with discs, forming a closed surface F^+. If F^+ has non-positive Euler characteristic, then it admits a one-vertex triangulation, in which each 1-cell is essential. (By a 'triangulation' here, all we mean is an expression of F^+ as union of 2-simplices with their edges identified in pairs.) If F^+ is a projective plane, then it admits a two-vertex triangulation. If F^+ is a sphere, then it admits a triangulation with three vertices. After subdividing, if necessary, each of these triangulations to increase the number of vertices, we obtain an ideal triangulation of F of the required form.

The following result is due to Thurston [17, Section 8].

Proposition 5.3. *There is a homotopy of* $f\colon F \to X$ *to a map which sends each ideal triangle of* $\mathrm{int}(F)$ *to a totally geodesic ideal triangle in* $\mathrm{int}(X)$.

Proof. We construct the homotopy on the 1-cells first. First pick a horoball neighbourhood N of the cusps of $\mathrm{int}(X)$ which is a union of disjoint copies of $S^1 \times S^1 \times [1, \infty)$. N lifts to a disjoint union of horoballs in $\mathbb{H}^3$. We may homotope f so that, after the homotopy, $f(F) \cap N$ is a union of vertical half-open annuli. Hence, for each (open) 1-cell α in the ideal triangulation, $\alpha - f^{-1}(\mathrm{int}(N))$ is a single interval. We may homotope this interval, keeping its endpoints fixed, to a geodesic in $\mathrm{int}(X)$. By assumption, this geodesic does not wholly lie in N. Hence, in the universal cover, this geodesic runs between distinct horoball lifts of N. We can then perform a further homotopy so that the whole of α is sent to a geodesic.

The boundary of each 2-cell of F^+ is a union of three 1-cells. The interior of each 1-cell is sent to a geodesic in $\mathrm{int}(X)$. By examining the universal cover of $\mathrm{int}(X)$, it is clear that we may map this 2-cell to an ideal triangle. Furthermore, the map of this 2-cell is homotopic to the original map, since the universal cover of $\mathrm{int}(X)$ is aspherical. q.e.d.

When F is in this form, it is an example of a 'pleated surface'. It inherits a metric, by pulling back the metric on $\mathrm{int}(X)$. This in fact gives $\mathrm{int}(F)$ a hyperbolic structure, since the metric arises from glueing a union of hyperbolic ideal triangles along their geodesic boundaries. Furthermore, this structure is complete, since the metric on $\mathrm{int}(X)$ is complete.

Proof of Theorem 5.1. We may use Proposition 5.3 to homotope f to a map g such that $g(\mathrm{int}(F))$ is a union of ideal triangles. Let N be the maximal horoball neighbourhood of T. Let N_- be a slightly smaller horoball neighbourhood of T, such that $g^{-1}(\partial N_-)$ is a disjoint union of simple closed curves, and so that $g(F)$ intersects ∂N_- transversely. We may find a sequence of such N_- whose union is the interior of N. We may also find a horoball neighbourhood N' of T, strictly contained in N_-, such that $g(F) \cap N'$ is a union of vertical half-open annuli. We will examine the intersection of $g(F)$ with the region $N_- - \mathrm{int}(N')$, which is a copy of $T^2 \times I$. Consider a component Y of $g^{-1}(N_- - \mathrm{int}(N'))$ for which $g(Y) \cap \partial N'$ is non-empty.

Claim 1. ∂Y contains no curve which bounds a disc in F.

Let C be such a curve, bounding a disc D in F. Then ∂D cannot map to $\partial N'$, since $[s] \in \pi_1(X)$ is non-trivial. Thus, the interior of D is disjoint from Y. If C intersected no 1-cells of F, then D would map into X in a totally geodesic fashion. Hence, $g(D)$ would lie in N_-, and so D would be Y. This is impossible, and so C must intersect some 1-cells of F. Hence, there is an arc in a 1-cell of F which is embedded D. But this 1-cell of F maps to a geodesic in X. This geodesic lifts to a geodesic in $\mathbb{H}^3$ which leaves and re-enters the same horoball lift of N_-. This cannot happen.

Claim 2. $g_*\colon \pi_1(Y) \to \pi_1(N_- - \mathrm{int}(N'))$ is injective.

Suppose that x is a non-zero element of $\pi_1(Y)$ which maps to $0 \in \pi_1(N - \mathrm{int}(N'))$. Then $g_*\colon \pi_1(Y) \to \pi_1(X)$ sends x to 0. But $\pi_1(Y) \to \pi_1(X)$ factors as $\pi_1(Y) \to \pi_1(F) \to \pi_1(X)$. The second of these maps is assumed to be injective. Therefore $\pi_1(Y) \to \pi_1(F)$ sends x to 0. This contradicts Claim 1.

Claim 3. Y is an annulus with one boundary component mapping to $\partial N'$ and the other mapping to ∂N_-.

Now, $\pi_1(N_- - \mathrm{int}(N')) \cong \mathbb{Z} \oplus \mathbb{Z}$. Hence, by Claim 2, Y must be a disc, a Möbius band or an annulus. However, if Y is not an annulus with one boundary component mapping to $\partial N'$ and the other mapping to ∂N_-, then F is a disc, a Möbius band or an annulus, which is a contradiction.

Claim 4. Each component of $g^{-1}(\text{int}(N))$ which touches $f^{-1}(T)$ is an open annulus in $\text{int}(F)$.

There is a sequence of horoball neighbourhoods N_- whose union is the interior of N. For each such N_- we showed in Claim 3 that each component Y of $g^{-1}(N_-)$ which touches $f^{-1}(T)$ is a half-open annulus in $\text{int}(F)$. The union of these half-open annuli is the required collection of open annuli in $\text{int}(F)$.

There is a standard homeomorphism which identifies N with $S^1 \times S^1 \times [1, \infty)$. We may pick such an identification so that $S^1 \times \{*\} \times \{1\}$ has slope s, where $\{*\}$ is some point in S^1. Consider now the covering

$$S^1 \times \mathbb{R} \times (1, \infty) \to S^1 \times S^1 \times (1, \infty) \cong \text{int}(N)$$

which is determined by the subgroup generated by $[s] \in \pi_1(T)$. Each open annulus from Claim 4 lifts to an open annulus in $S^1 \times \mathbb{R} \times (1, \infty)$. Now, $S^1 \times \{0\} \times (1, \infty)$ inherits a hyperbolic structure from $\text{int}(N)$, which makes it isometric to a 2-dimensional horocusp. Let the map $p \colon S^1 \times \mathbb{R} \times (1, \infty) \to S^1 \times \{0\} \times (1, \infty)$ be orthogonal projection onto this submanifold. Note that p need not respect the product structure of $S^1 \times \mathbb{R} \times (1, \infty)$. Each open annulus of Claim 4 is mapped surjectively onto $S^1 \times \{0\} \times (1, \infty)$. Also, since p does not increase distances, the area A' that each open annulus inherits from $S^1 \times \{0\} \times (1, \infty)$ is no more than the area which it inherits from $\text{int}(N)$. However, A' is at least $l(s)$, since this is the area of the 2-dimensional horocusp $S^1 \times \{0\} \times (1, \infty)$. Thus, the hyperbolic area of F is more than $l(s)\,|f(F) \cap T|$. But the Gauss-Bonnet formula [3, Proposition B.3.3] states that its total area is $-2\pi\chi(F)$. We therefore deduce that

$$l(s)\,|f(F) \cap T| < -2\pi\,\chi(F). \qquad \text{q.e.d.}$$

6. 'Almost hyperbolic' 3-manifolds are hyperbolic

Throughout this paper, we have studied 3-manifolds which have a complete finite volume negatively curved Riemannian metric. It is a major conjecture whether the existence of such a metric on a 3-manifold implies the existence of a complete finite volume hyperbolic structure. In this section, we provide evidence for this conjecture by considering 3-manifolds which are 'almost hyperbolic' in the following sense.

Definition 6.1. Let δ be a positive real number. Let M be a compact orientable 3-manifold with ∂M a (possibly empty) collection of tori. Let g be a Riemannian metric on int(M). Then (M,g) is δ-*pinched* if

$$-1-\delta \leq \kappa_{\inf}(M,g) \leq \kappa_{\sup}(M,g) \leq -1+\delta.$$

We say that M is *almost hyperbolic* if, for all $\delta > 0$, there a δ-pinched complete finite volume Riemannian metric on its interior.

The main theorem of this section is the following result.

Theorem 6.2. *Let M be a compact orientable 3-manifold with ∂M a (possibly empty) union of tori. If M is almost hyperbolic, then it has a complete finite volume hyperbolic structure.*

We proved this theorem in the course of proving several other results in this paper. We recently discovered that it has also been proved by Zhou [18] using methods similar to our own (but not identical). It was also known to Petersen [14]. However, since the journal [18] is relatively poorly circulated in the West, it seems worthwhile to include a summary of our proof of this result here.

The idea behind our proof of this theorem is as follows. Since M is almost hyperbolic, there is a sequence of positive real numbers δ_i tending to zero, and complete finite volume Riemannian metrics g_i on int(M) such that (M,g_i) is δ_i-pinched. We show that some subsequence 'converges' to a 'limit' manifold (M_∞, g_∞) which is a 3-manifold M_∞ with a complete finite volume hyperbolic hyperbolic metric g_∞. Hence, M_∞ is the interior of some compact orientable 3-manifold $\overline{M}_\infty$ with $\partial\overline{M}_\infty$ a (possibly empty) union of tori. If $\overline{M}_\infty$ is homeomorphic to M, then we have found a complete finite volume hyperbolic structure on M. There is no immediate reason why $\overline{M}_\infty$ should be homeomorphic to M, but we will show that, if it is not, then there exist slopes $(s_i^1,\ldots,s_i^{n(i)})$ on $\partial\overline{M}_\infty$ such that $\overline{M}_\infty(s_i^1,\ldots,s_i^{n(i)})$ is homeomorphic to M. The length of each of these slopes tends to infinity as $i\to\infty$. Lemma 4.2 then gives us a contradiction.

In Section 4, we exploited the well-known theory of convergent sequences of hyperbolic manifolds, and in that case, non-trivial convergence corresponds to hyperbolic Dehn surgery [3]. In the case here, the infinite sequence of manifolds are not hyperbolic, merely negatively curved, but a similar theory applies. We recall the following definition [8], due to Gromov (see also [6]).

Definition 6.3. Let M_1 and M_2 be two metric spaces, with metrics d_1 and d_2 respectively, and basepoints x_1 and x_2. If ϵ is a positive real number, then an *ϵ-approximation* between (M_1, d_1, x_1) and (M_2, d_2, x_2) is a relation $R \subset M_1 \times M_2$ such that

(i) the projections $p_1 \colon R \to M_1$ and $p_2 \colon R \to M_2$ are both surjections,

(ii) if xRy and $x'Ry'$, then $|d_1(x, x') - d_2(y, y')| < \epsilon$, and

(iii) $x_1 R x_2$.

If x is a point in a metric space (M, d) and r is a positive real number, we denote the ball of radius r about x by $B_M(x, r)$. If we wish to emphasise the metric on M, we may also write $B_{(M,d)}(x, r)$. If $(M_\infty, d_\infty, x_\infty)$ and $\{(M_i, d_i, x_i) \colon i \in \mathbb{N}\}$ are metric spaces with basepoints, then we say that the sequence (M_i, d_i, x_i) *converges* to $(M_\infty, d_\infty, x_\infty)$ if, for all $r > 0$, there is a sequence of positive real numbers $\epsilon_i \to 0$ and ϵ_i-approximations between $B_{M_i}(x_i, r)$ and $B_{M_\infty}(x_\infty, r)$. In this case, we write $(M_i, d_i, x_i) \to (M_\infty, d_\infty, x_\infty)$.

The following example will be useful. Its proof (which is omitted) is a straightforward application of Jacobi fields.

Lemma 6.4. *Let M_i be a sequence of simply-connected n-manifolds with complete Riemannian metrics g_i such that $\kappa_{\sup}(M_i, g_i) \to -1$ and $\kappa_{\inf}(M_i, g_i) \to -1$. Let x_i be a basepoint in M_i, and let x_∞ be any point in hyperbolic n-space $\mathbb{H}^n$. Let r be any positive real number. Then for i sufficiently large, there is a sequence of real numbers $k_i > 1$ tending to 1, and a sequence of k_i-bi-Lipschitz homeomorphisms $h_i \colon B_{M_i}(x_i, r) \to B_{\mathbb{H}^n}(x_\infty, r)$. In particular, (M_i, g_i, x_i) converges to $\mathbb{H}^n$ with basepoint x_∞.*

The following theorem of Gromov is a uniqueness result for convergent sequences. A simple proof of this result can be found in [8].

Theorem 6.5. [8] *Let (M_i, d_i, x_i) be a sequence of complete metric spaces with basepoints, such that every closed ball in M_i is compact. Suppose that there are complete pointed metric spaces (M, d, x) and (M', d', x') such that*

$$(M_i, d_i, x_i) \to (M, d, x)$$
$$(M_i, d_i, x_i) \to (M', d', x').$$

Then (M, d, x) and (M', d', x') are isometric pointed metric spaces.

Vital in our construction of a hyperbolic metric on int(M) will be the following theorem.

Theorem 6.6. [8] *Let (M_i, d_i, x_i) be a sequence of complete metric spaces (with basepoints) in which bounded balls are compact. Then the following are equivalent.*

(1) There is a subsequence $\{(M_j, d_j, x_j) \colon j \in J \subset \mathbb{N}\}$ converging to a complete metric space (M, d, x).

(2) There is a subsequence $\{(M_k, d_k, x_k) \colon k \in K \subset \mathbb{N}\}$ such that for all $\epsilon > 0$ and $r > 0$, there is a natural number $K(r, \epsilon)$ with the property that the number of ϵ-balls required to cover $B_{M_k}(x_k, r)$ is less than $K(r, \epsilon)$.

Gromov also proved that, under certain circumstances, convergence in the above sense implies bi-Lipschitz convergence. The proof of this result [5] readily gives the following theorem.

Theorem 6.7. *Let (M_i, g_i, x_i) be a sequence of Riemannian manifolds converging to some space $(M_\infty, d_\infty, x_\infty)$. Let r be a positive real number. Suppose that $\kappa_{\sup}(B_{M_i}(x_i, r), g_i)$ and $\kappa_{\inf}(B_{M_i}(x_i, r), g_i)$ are bounded above and below by constants independent of i. Suppose also that the injectivity radius of $B_{M_i}(x_i, r)$ is bounded below by a constant independent of i. Then for sufficiently large i, there is a sequence of real numbers $k_i > 1$ tending to 1, a sequence of positive real numbers ϵ_i tending to zero and a sequence of k_i-bi-Lipschitz homeomorphisms $h_i \colon B_{M_\infty}(x_\infty, r) \to U_i$, where $B_{M_i}(x_i, r - \epsilon_i) \subset U_i \subset B_{M_i}(x_i, r + \epsilon_i)$.*

The following lemma follows immediately from Theorem 6.6.

Lemma 6.8. *Let (M_i, g_i, x_i) be a sequence of δ_i-pinched complete Riemannian n-manifolds with $\delta_i \to 0$. Then some subsequence converges to a pointed metric space $(M_\infty, d_\infty, x_\infty)$.*

Proof. Let $(\tilde{M}_i, \tilde{g}_i)$ be the universal cover of (M_i, g_i) and let $\tilde{x}_i \in \tilde{M}_i$ be a lift of the basepoint x_i. Then Lemma 6.4 states that $(\tilde{M}_i, \tilde{g}_i, \tilde{x}_i)$ converges. Hence, Theorem 6.6 states that we may pass to a subsequence $(\tilde{M}_k, \tilde{g}_k, \tilde{x}_k)$ so that for any $\epsilon > 0$ and $r > 0$, there is a natural number $K(r, \epsilon)$ such that the number of ϵ-balls required to cover $B_{\tilde{M}_k}(\tilde{x}_k, r)$ is less than $K(r, \epsilon)$. Such a covering projects to a covering of $B_{M_k}(x_k, r)$ by ϵ-balls. Hence, some subsequence (M_j, g_j, x_j) converges. q.e.d.

So, in our case where M_i is a fixed manifold M, some subsequence (M, g_i, x_i) converges to a limit $(M_\infty, d_\infty, x_\infty)$. In general, we lose a great deal of information when passing from (M, g_i) to (M_∞, d_∞). In particular, (M_∞, d_∞) need not be a hyperbolic 3-manifold. However, if we pick the basepoints x_i judiciously, then M_∞ will be hyperbolic. We shall pick x_i in the 'thick' part of (M, g_i). If ϵ is a positive real number, then we denote the ϵ-thick part of (M, g_i) by $(M, g_i)_{[\epsilon,\infty)}$ and ϵ-thin part of (M, g_i) by $(M, g_i)_{(0,\epsilon]}$.

The Margulis lemma [3] describes the ϵ-thin part of hyperbolic manifolds for ϵ sufficiently small. There is an extension of this result to negatively curved Riemannian manifolds [2]. This implies that there is a positive real number μ (called a Margulis constant) with the following property. If M is an orientable 3-manifold and g is a δ-pinched Riemannian metric on M with $\delta < 1$, then each component X of $(M, g)_{(0,\epsilon]}$ for $\epsilon \leq \mu$ is diffeomorphic to one of the following possibilities.

(i) $X \cong D^2 \times S^1$. In this case, X is known as a 'tube'. It is a neighbourhood of a closed geodesic in M with length less than ϵ.

(ii) $X \cong S^1$. Then X is a closed geodesic with length precisely ϵ. By perturbing our choice of ϵ a little, we can ensure that this possibility never arises.

(iii) $X \cong S^1 \times S^1 \times [0, \infty)$. Then X is a 'cusp'.

In fact, a closer examination of the metric on the μ-thin part of (M, g) readily yields the following two results.

Proposition 6.9. *There is a function $D\colon (0, \mu/2) \to \mathbb{R}_+$ with $D(\epsilon) \to \infty$ as $\epsilon \to 0$, and having the following property. Pick real numbers δ and ϵ with $0 < \delta < 1$ and $0 < \epsilon < \min\{1, \mu/2\}$. If M is a compact orientable 3-manifold with a complete finite volume δ-pinched Riemannian metric g on its interior, then the distance between $(M, g)_{(0,2\epsilon^2]}$ and $(M, g)_{[2\epsilon,\infty)}$ is at least $D(\epsilon)$.*

Proposition 6.10. *There is a function $H\colon (0, \mu) \to \mathbb{R}_+$ with $H(\epsilon) \to \infty$ as $\epsilon \to 0$ and having the following property. Pick $\delta \in (0, 1)$. Let M be a compact orientable 3-manifold with a complete finite volume δ-pinched Riemannian metric g on its interior. If X is a component of $(M, g)_{(0,\mu]}$ which is a neighbourhood of a geodesic of length at most ϵ, then the length of a meridian curve on ∂X is at least $H(\epsilon)$.*

We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2. Since M is almost hyperbolic, there is a sequence of positive real numbers δ_i tending to zero, and complete finite volume Riemannian metrics g_i on $\text{int}(M)$, such that (M, g_i) is δ_i-pinched. We may assume that, for all i, $\delta_i < 1$. For each i, pick a basepoint x_i in $(M, g_i)_{[\mu,\infty)}$, where μ is the constant mentioned above.

Claim 1. Some subsequence (M, g_j, x_j) converges to $(M_\infty, g_\infty, x_\infty)$, which is a 3-manifold M_∞ with a complete hyperbolic Riemannian metric g_∞.

Some subsequence converges to a complete metric space $(M_\infty, d_\infty, x_\infty)$ by Lemma 6.8. Pass to this subsequence. Let y be any point in M_∞. We wish to examine a neighbourhood of y.

Let r be $d_{M_\infty}(x_\infty, y) + \mu$. By definition, there is a sequence $\epsilon_i \to 0$ and ϵ_i-approximations between $(B_{(M,g_i)}(x_i, r), g_i, x_i)$ and $(B_{M_\infty}(x_\infty, r), d_\infty, x_\infty)$. From this, we get a sequence of ϵ_i-approximations between $(B_{(M,g_i)}(x_i, r), g_i, y_i)$ and $(B_{M_\infty}(x_\infty, r), d_\infty, y)$ for some $y_i \in M$. For any real number z with $\epsilon_i < z < \mu$, each ϵ_i-approximation restricts to an ϵ_i-approximation between (U_i, g_i, y_i) and $(B_{M_\infty}(y, z), d_\infty, y)$, where $B_{(M,g_i)}(y_i, z - \epsilon_i) \subset U_i \subset B_{(M,g_i)}(y_i, z + \epsilon_i)$. We can extend this to a $2\epsilon_i$-approximation between $(B_{(M,g_i)}(y_i, z), g_i, y_i)$ and $(B_{M_\infty}(y, z), d_\infty, y)$. So,

$$(B_{(M,g_i)}(y_i, z), g_i, y_i) \to (B_{M_\infty}(y, z), d_\infty, y).$$

If we insist that $\epsilon_i < \epsilon \leq \mu/2$, then

$$x_i \in (M, g_i)_{[\mu,\infty)} \subset (M, g_i)_{[2\epsilon,\infty)}.$$

By construction, $y_i \in B_{(M,g_i)}(x_i, r)$. For ϵ sufficiently small, the distance between $(M, g_i)_{(0,2\epsilon^2]}$ and $(M, g_i)_{[2\epsilon,\infty)}$ is more than r, by Proposition 6.9. Hence, $y_i \in (M, g_i)_{[2\epsilon^2,\infty)}$. Thus, $B_{(M,g_i)}(y_i, \epsilon^2)$ is isometric to a ball of radius ϵ^2 in the universal cover $(\tilde{M}, \tilde{g}_i)$ of (M, g_i). But $(\tilde{M}, \tilde{g}_i)$ is a complete simply-connected Riemannian 3-manifold, and both $\kappa_{\text{sup}}(\tilde{M}, \tilde{g}_i)$ and $\kappa_{\text{inf}}(\tilde{M}, \tilde{g}_i)$ tend to -1 as $i \to \infty$. Thus, Lemma 6.4 states that $B_{(M,g_i)}(y_i, \epsilon^2)$ converges to a ball of radius ϵ^2 in $\mathbb{H}^3$, with basepoint at the centre p of the ball. Theorem 6.5 implies that this ball is isometric to $B_{M_\infty}(y, \epsilon^2)$, via an isometry taking p to y. Thus, (M_∞, d_∞) is a 3-manifold with a complete hyperbolic Riemannian metric g_∞. This proves the claim.

Claim 2. Let r be any positive real number. For i sufficiently large, there is a sequence of real numbers $k_i > 1$ tending to 1, a sequence of positive real numbers ϵ_i tending to zero and a sequence of k_i-bi-Lipschitz homeomorphisms

$$h_i \colon B_{M_\infty}(x_\infty, r) \to U_i,$$

where

$$B_{(M,g_i)}(x_i, r - \epsilon_i) \subset U_i \subset B_{(M,g_i)}(x_i, r + \epsilon_i).$$

We just need to check that the conditions of Theorem 6.7 are satisfied. The restrictions on $\kappa_{\sup}(B_{(M,g_i)}(x_i, r), g_i)$ and $\kappa_{\inf}(B_{(M,g_i)}(x_i, r), g_i)$ hold automatically since (M, g_i) is δ_i-pinched with $\delta_i \to 0$. Since $x_i \in (M, g_i)_{[\mu,\infty)}$, Proposition 6.9 implies that there is some $\epsilon \le \mu/2$ such that $(M, g_i)_{[2\epsilon^2,\infty)} \supset B_{(M,g_i)}(x_i, r)$ for all i sufficiently large. Thus Theorem 6.7 proves the claim.

Claim 3. (M_∞, g_∞) has finite volume.

In the proof of Proposition 3.3, we established that

$$|M|\pi/2 \ge (-\kappa_{\sup}(M, g_i))^{3/2}\mathrm{Vol}(M, g_i).$$

Since $\kappa_{\sup}(M, g_i) \to -1$, we deduce that the sequence $\mathrm{Vol}(M, g_i)$ is bounded above. Now,

$$\mathrm{Vol}(M_\infty, g_\infty) = \lim_{r\to\infty} \mathrm{Vol}(B_{M_\infty}(x_\infty, r), g_\infty),$$

and, using the notation in Claim 2,

$$\begin{aligned}\mathrm{Vol}(B_{M_\infty}(x_\infty, r), g_\infty) &\le (k_i)^3\mathrm{Vol}(B_{(M,g_i)}(x_i, r + \epsilon_i), g_i)\\ &\le (k_i)^3\mathrm{Vol}(M, g_i).\end{aligned}$$

Thus, $\mathrm{Vol}(M_\infty, g_\infty)$ is finite.

This implies that $M_\infty = \mathrm{int}(\overline{M}_\infty)$ for some compact orientable 3-manifold $\overline{M}_\infty$, with $\partial\overline{M}_\infty$ a (possibly empty) union of tori.

Claim 4. Let ϵ be a positive real number less than μ such that $(M_\infty, g_\infty)_{(0,\epsilon]}$ is either empty or consists only of horoball neighbourhoods of cusps. Then, for i sufficiently large, there is a sequence of real numbers $k_i' > 1$ tending to 1, and k_i'-bi-Lipschitz homeomorphisms $h_i'\colon (M_\infty, g_\infty)_{[\epsilon,\infty)} \to (M, g_i)_{[\epsilon,\infty)}$.

We pick $r > 0$ so that $B_{M_\infty}(x_\infty, r) \supset (M_\infty, g_\infty)_{[\epsilon,\infty)}$. Let T be the boundary of $(M_\infty, g_\infty)_{[\epsilon,\infty)}$ which is a collection of tori. Using the notation in Claim 2, the homeomorphism $h_i\colon B_{M_\infty}(x_\infty, r) \to U_i$ is almost an isometry for i large. Therefore for large i, there is a sequence of positive real numbers γ_i tending to zero, such that $h_i(T)$ separates $(M, g_i)_{(0,\epsilon-\gamma_i]}$ from $(M, g_i)_{[\epsilon+\gamma_i,\infty)}$. But, using the Margulis Lemma for negatively curved 3-manifolds, $(M, g_i)_{[\epsilon-\gamma_i,\epsilon+\gamma_i]}$ is homeomorphic to a collection of copies of $T^2 \times I$. Hence, $h_i(T)$ is isotopic to the boundary of $(M, g_i)_{[\epsilon,\infty)}$, since any torus in $T^2 \times I$ which separates the boundary components is isotopic to either boundary component. We may therefore modify h_i to h'_i: $(M_\infty, g_\infty)_{[\epsilon,\infty)} \to (M, g_i)_{[\epsilon,\infty)}$, ensuring that the h'_i are bi-Lipschtiz homeomorphisms as claimed.

Claim 5. Let ϵ be a positive real number less than μ such that $(M_\infty, g_\infty)_{(0,\epsilon]}$ is either empty or consists only of horoball neighbourhoods of cusps. Then the length of the core geodesic of each tube component of $(M, g_i)_{(0,\epsilon]}$ tends to zero, as $i \to \infty$.

If not, we may find a positive real number $\alpha \leq \epsilon$ and a subsequence in which $(M, g_i)_{(0,\epsilon]}$ contains a geodesic of length at least α. Applying Claim 4, we find that, for i sufficiently large, there is a sequence of real numbers $k''_i > 1$ tending to 1, and k''_i-bi-Lipschitz homeomorphisms h''_i: $(M_\infty, g_\infty)_{[\alpha,\infty)} \to (M, g_i)_{[\alpha,\infty)}$. In particular, there is a geodesic in (M_∞, g_∞) of length at most $\alpha k''_i$. But $\alpha \leq \epsilon$, which is less than the length of the shortest geodesic in M_∞. This is a contradiction, which proves the claim.

Fix $\epsilon \leq \mu$ such that $(M_\infty, g_\infty)_{(0,\epsilon]}$ is either empty or consists only of horoball neighbourhoods of cusps. Now, $(M, g_i)_{(0,\epsilon]}$ is a (possibly empty) collection of tubes and a (possibly empty) collection of cusps. If, for infinitely many i, $(M, g_i)_{(0,\epsilon]}$ contains no tubes, then Claim 4 implies that M is homeomorphic to $\overline{M}_\infty$. By Claims 1 and 3, M_∞ has a complete finite volume hyperbolic structure, which proves the theorem in this case.

Consider now the case where $(M, g_i)_{(0,\epsilon]}$ is, for infinitely many i, a collection of tubes $X_i^1, \dots X_i^{n(i)}$ and possibly also some cusps. Claim 4 implies that, for each i sufficiently large, the meridian slope on X_i^j corresponds to a slope s_i^j on $\partial\overline{M}_\infty$, and that M is homeomorphic to $\overline{M}_\infty(s_i^1, \dots, s_i^{n(i)})$. By passing to a subsequence, we may ensure that $n(i)$ is some fixed integer n, and that, for each j, the slopes s_i^j all lie a fixed torus T_j. Claim 5 states that the core geodesic of X_i^j tends to zero as $i \to \infty$. Hence, Proposition 6.10 states that the length of the meridian slope on X_i^j tends to infinity. The length of s_i^j on $(M_\infty, g_\infty)_{[\epsilon,\infty)}$ differs from length of the corresponding meridian slope on X_i^j by a factor of at most k'_i, which converges to 1, as in Claim 4. Thus, $l(s_i^j) \to \infty$ as $i \to \infty$. Therefore, by passing to a subsequence, we may assume that the slopes $s_i^j \neq s_k^j$ if $i \neq k$. Lemma 4.2 gives us a contradiction. q.e.d.

Remark. The proof of Theorem 6.2 actually gives something a little stronger. It shows that if (M, g_i) is a sequence of δ_i-pinched Riemannian manifolds with $\delta_i \to 0$, then int(M) has a complete finite volume hyperbolic metric h, and (for all i sufficiently large) there is a sequence of real number $k_i > 1$, tending to 1, and a sequence of k_i-bi-Lipschitz homeomorphisms between (M, g_i) and (M, h).

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