ELSEVIER

Topology and its Applications 102 (2000) 239–252

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Immersed, virtually-embedded, boundary slopes ☆

M. Baker [a,1], D. Cooper [b,*]

[a] *L'Institut Mathématique Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France*
[b] *Department of Mathematics, University of California, Santa Barbara, CA 93106, USA*

Received 23 March 1998; received in revised form 14 September 1998

Abstract

For the figure eight knot, we show that slopes with even numerator are slopes of immersed surfaces covered by incompressible, boundary-incompressible embeddings in some finite cover. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Knot; Immersed boundary slope

AMS classification: 57M25; 57M50

1. Introduction

Definition 1.1. A *slope* on a torus, T, is the isotopy class of an essential un-oriented simple closed curve, α, on T. Suppose that X is a three manifold with a boundary component which is a torus T. An *immersed boundary slope* on T is a slope, α, on T such that there is a proper immersion of a compact, connected, oriented, surface into X which is π_1-injective and which is an embedding in a neighborhood of the boundary of X. We also require that the surface cannot be homotoped into the boundary of X by a proper homotopy. The boundary of the surface consists of loops on T parallel to α. If the immersion is an embedding then we also call the slope an *embedded boundary slope.* If the immersion is covered by an embedding in some finite cover, then we also call the slope a *virtually embedded boundary slope*.

It is known that a knot has only finitely many *embedded* boundary slopes [5]. It is easy to see that a torus knot has only two immersed boundary slopes, and these are also embedded boundary slopes. Several examples of immersed boundary slopes which are

☆ Research supported by, and done at, Université de Rennes 1 and IHP.
* Corresponding author. Email: cooper@math.ucsb.edu.
1 Email: mark.baker@univ-rennes1.fr.

PII: S0166-8641(98)00152-7

not embedded boundary slopes have been constructed. Some immersed boundary slopes have been found for the figure eight knot [7]. It was first shown in [1] that a compact 3-manifold with boundary a single torus may have infinitely many immersed boundary slopes by exhibiting a large family of once punctured torus bundles over S^1 with infinitely many virtually embedded boundary slopes. It is shown in [13] that there is a compact 3-manifold with torus boundary such that every slope is an immersed boundary slope, though it is not known if they are virtually embedded boundary slopes. Recently Joseph Maher [11] has used Theorem 1.4 to show that every hyperbolic 2-bridge knot and every hyperbolic punctured torus bundle has the property that every slope is a virtually embedded boundary slope.

Proposition 1.2. *Suppose that X is a compact Seifert fibered three-manifold with boundary an incompressible torus T. Then there are only two immersed boundary slopes. These are the slope of a regular fiber and the slope which is* 0 *in $H_1(X)$. These are both embedded boundary slopes. There is an essential embedded vertical annulus with slope the regular fiber. In particular the only immersed boundary slopes of the exterior of the (p,q) torus knot are* 0 *and pq and these are both embedded boundary slopes.*

In this paper we construct many virtually embedded boundary slopes in the figure eight knot exterior, in particular:

Theorem 1.3. *Every slope is a virtually embedded boundary slope for the two-fold cover of the figure eight knot exterior.*

The main tool for this is the following general result which gives conditions under which homology classes in finite covers give rise to virtually embedded boundary slopes.

Theorem 1.4. *Let M be a compact, connected, orientable, atoroidal and irreducible* 3*-manifold with boundary a finite number of tori. Suppose that S is a compact, connected, non-separating, orientable, incompressible surface properly embedded in M which is not a fiber of a fibration of M. Also suppose that ∂S contains some components with slope α, on a torus, T, in the boundary of M. Then α is a virtually embedded boundary slope.*

Moreover there is a finite cyclic cover, $\pi : \widetilde{M} \to M$ dual to S and a compact, connected, orientable, incompressible, boundary-incompressible, surface F, properly embedded in $\widetilde{M}$. The boundary of F consists of a non-empty set of essential, parallel curves lying on some component, $\widetilde{T}$, of $\partial\widetilde{M}$ which covers T. Also $\pi|F$ is an immersion which is an embedding in a neighborhood of the boundary, and the boundary is mapped to loops parallel to α.

This theorem provides a method for finding virtually embedded boundary slopes. One constructs a finite cover, $\widetilde{X}$, of the knot exterior X. Then one determines the kernel of the map

$$incl_* : H_1(\partial\widetilde{X}) \to H_1(\widetilde{X}).$$

An element, β, of the kernel is the boundary of a compact orientable incompressible surface, S, in $\widetilde{X}$. This surface may be chosen to be non-separating if β is a primitive element of the kernel. To apply the theorem, we need to know that S is not a fiber of some fibration of $\widetilde{X}$. This can be guaranteed if S is disjoint from at least one component of $\partial\widetilde{X}$, since it is clear that a fiber must meet every boundary component. Thus one has the homological problem of finding elements, β, in the kernel which also lie in the homology of the boundary minus some torus. Then β is a sum of loops, each on one of the boundary tori, and one of these loops is chosen as α. If the surface S is not connected, one may use one of the components of S. In order to obtain a surface in X which is embedded in a neighborhood of ∂X one also needs that α projects one-to-one into ∂X.

The idea for proving Theorem 1.4 is that of [3]. One takes two lifts, $\widetilde{S}_0$ and $\widetilde{S}_n$, of S to $\widetilde{M}$ which are "far apart". Then one connects pairs of boundary components on $\widetilde{S}_0$ and $\widetilde{S}_n$ by boundary parallel tubes to obtain a closed, embedded, surface H in $\widetilde{M}$. The techniques of [3] can be extended to show that H is incompressible. One now deletes from H an essential boundary-parallel annulus and pushes the boundary of the new surface into $\partial\widetilde{M}$. This is the incompressible surface F. For the sake of variety, we will give a somewhat different proof that H is incompressible.

Proof of Proposition 1.2. Since X is Seifert fibered with non-empty boundary there is a finite cyclic cover, $\widetilde{X}$, of X on which the induced Seifert fibering is a circle bundle $p:\widetilde{X}\to T_0$ over a compact surface, T_0, with one boundary component. Since ∂X is incompressible it follows that T_0 is not a disc. Thus it suffices to show that the only immersed boundary slopes in this circle bundle are a fiber and a *longitude*, i.e., an essential curve in $\partial\widetilde{X}$ which is zero in $H_1(\widetilde{X},\mathbb{Q})$.

Let $\theta:F\to\widetilde{X}$ be a proper π_1-injective map which is not homotopic rel ∂F into $\partial\widetilde{X}$. Now $K=\mathit{ker}[p_*:\pi_1\widetilde{X}\to\pi_1T_0]\cong\pi_1S^1$ is a normal cyclic subgroup of $\pi_1(\widetilde{X})$. We will identify π_1F with the subgroup $\theta_*\pi_1F$ of $\pi_1(\widetilde{X})$. The intersection of $\mathit{ker}\ p_*$ with π_1F is a cyclic normal subgroup, H, of π_1F. Since F is an orientable surface with non-empty boundary, either H is trivial or F is an annulus. Thus if H is non-trivial then F is an annulus whose fundamental group intersects K in a non-trivial subgroup. Now π_1F is generated by a boundary component, C, of F. Also C is a loop on $\partial\widetilde{X}$ and some power of C is in H. Thus some power of C is freely homotopic in $\widetilde{X}$ to a power of a fiber. By considering the action of C on the universal cover one sees that C is a homotopic to a fiber in $\partial\widetilde{X}$. In the remaining case that H is trivial then $p\circ\theta:F\to T_0$ is a proper π_1-injective map. Therefore it is either homotopic rel ∂F into ∂T_0 or is homotopic to a finite covering. In the first case this homotopy is covered by a homotopy of F rel ∂F into $\partial\widetilde{X}$ which is not allowed. Hence $p\circ\theta$ is homotopic to a covering and therefore has non-zero degree. Thus $\theta_*[\partial F]$ is non-zero in $H_1(\partial\widetilde{X})$. This class is the boundary of $\theta_*[F]$. It follows that the boundary of F consists of longitudes. □

Question 1.5. Is every immersed boundary slope also a virtually embedded boundary slope?

Question 1.6. If M is a compact 3-manifold with boundary a torus and if M is atoroidal and not a Seifert fiber space, is every boundary slope an immersed boundary slope?

In order to apply the theorem more generally one needs to know if the following is possible:

Question 1.7. Suppose that X is a compact, atoroidal manifold with boundary a torus T. Suppose that X is not a Seifert fiber space. Is it possible that there is an essential simple closed curve α on T such that for every finite cover $\widetilde{X} \to X$ and every closed curve $\widetilde{\alpha}$ in $\widetilde{X}$ which projects to some non-zero multiple of α, that $[\widetilde{\alpha}] = 0$ in $H_1(\widetilde{X}, \mathbb{Q})$. In particular, can this happen for a knot exterior with α a longitude?

2. Product regions

Definition 2.1. Let N be a connected 3-manifold with connected boundary. Suppose that $\partial_v N$ is a compact subsurface of ∂N such that each component is an annulus. Suppose that $\partial_v N$ separates ∂N into two incompressible components with closure S_0, S_1 and which are diffeomorphic. Also suppose that each annulus has one boundary component in each of S_0 and S_1. We call $\partial_v N$ the *vertical* boundary of N, and $\partial_h = S_0 \cup S_1$ the *horizontal boundary*. We call N a *relative cobordism* between S_0 and S_1. A *vertical arc* is an arc in N with one endpoint in each of S_0 and S_1. A *vertical square* is a disc D properly embedded in N which intersects $\partial_v N$ in two vertical arcs which are called the *vertical boundary* of D and written $\partial_v D$. The *horizontal boundary* of D is $\partial_h D = D \cap \partial_h N$. A vertical square is called *essential* if it is not isotopic rel boundary into the boundary of N. An annulus embedded in N is called *vertical* if it has one boundary component in each of S_0 and S_1. We denote the unit interval as $I = [0, 1]$. A *product region* for N is a submanifold $\Phi \times I$ of N such that Φ is a compact, possibly disconnected, surface and for $i = 0, 1$ we have $\Phi \times i$ is an incompressible subsurface of S_i. We also require that $\Phi \times I$ contains $\partial_v N$, and that each component of $\Phi \times I$ intersects $\partial_v N$.

In the above definition we do not assume that N is compact. In the application N will be a compact manifold minus some components of its boundary. Our first goal is to now show that there is a maximal product region which is unique up to isotopy. This assertion follows from the existence and properties of the characteristic submanifold of N. We will give a direct proof instead of deducing it from standard statements: [10,8]. Also compare [3].

Lemma 2.2. *Suppose that N is an orientable, irreducible relative cobordism and D is a vertical square in N. If A is an incompressible vertical annulus properly embedded in N, then we may isotope D rel $\partial_v D$ so that $D \cap A$ consists of vertical arcs. If D' is another vertical square in N and if $\partial_v D$ and $\partial_v D'$ are disjoint then we may isotope D rel $\partial_v D$ so that $D \cap D'$ consists of vertical arcs.*

Proof. Since $\partial A = \partial_h A$ we may isotope D rel $\partial_v D$ so that A and D are transverse and their intersection has the smallest number of components. Suppose C is a component of $A \cap D$. If C is a circle then C bounds a disc $D_1 \subset D$. Since A is incompressible C also bounds a disc $\Delta \subset A$. Choose C innermost on A then the union of these discs is a sphere $D_1 \cup \Delta$, which bounds a ball, B, because N is irreducible. The interior of B is disjoint from D. Thus the disc D_1 can be isotoped across B (without hitting D) to remove C. Repeating this, gives an isotopy of D fixed on the boundary, which removes all circle components.

If C is an arc with both endpoints on S_0 then there is a disc Δ in A with boundary C plus an arc, α, in ∂A. We may choose an innermost such disc so that the interior of Δ is disjoint from D. There is also a disc Δ' in D with boundary C plus an arc, α', in ∂D. Then $\alpha \cup \alpha'$ is a loop in S_0 which bounds the disc $\Delta \cup \Delta'$ and, since S_0 is incompressible, this loop bounds a disc in S_0. We may choose $\alpha \cup \alpha'$ to be an innermost loop in S_0. But then C can be removed by an isotopy of D. Similarly arcs with both endpoints in S_1 can be removed. Hence $A \cap D$ consists of vertical arcs.

A similar argument works for a disc D' in place of the annulus A. $\square$

Theorem 2.3. *Suppose that N is a relative cobordism between S_0 and S_1 and assume that N is orientable and irreducible. Then there is a product region, $\Phi \times I$ in N such that every vertical square in N can be properly isotoped into $\Phi \times I$.*

Proof. Observe that a regular neighborhood of $\partial_v N$ is a product region for N. Consider a product region $\Phi \times I$ of N with minimal Euler-characteristic. We will regard Φ as a subsurface of S_0. Observe that since every component of Φ contains a component of ∂S_0, no component of Φ is a disc. In particular, since Φ is incompressible it follows that $\chi(\Phi) \geqslant \chi(S_0)$ and so there is such a Φ with minimal Euler characteristic. Observe that $(\partial\Phi) \times I$ is a collection of vertical annuli. Now given a vertical square D, we can isotope it so that the intersection of D with $(\partial\Phi) \times I$ consists of vertical arcs, and has the minimum number of such arcs. Since a product region must contain $\partial_v N$ and since D contains two vertical arcs in its boundary, it follows that either D is contained in $\Phi \times I$ or else $\partial_h D$ intersects $\partial\Phi$ in the interior of N. In the latter case, there is an arc, γ, in $\partial_h D$ with endpoints in $\partial\Phi$ and interior disjoint from Φ. Then γ is not isotopic rel endpoints into Φ for otherwise we could isotope D to reduce the number of vertical arcs of intersection of D with A. Let δ_1, δ_2 be the two vertical arcs in $D \cap A$ which have endpoints in common with γ. Then there is a sub-disc D_- of D with boundary $\delta_1 \cup \gamma \cup \delta_2 \cup \gamma'$ where γ' is contained in $D \cap S_1$.

Define R to be a regular neighborhood of $\Phi \times I \cup D_-$. Then $R = \Psi \times I$ where Ψ is a regular neighborhood of $\Phi \cup \gamma$ in S_0. Since γ is not isotopic rel endpoints into Φ, it follows that Ψ is incompressible thus R is a product region with lower Euler characteristic, a contradiction. $\square$

We shall call such a product region a *maximal product region*. This terminology will be justified soon.

Theorem 2.4. *Suppose that N is a relative cobordism between S_0 and S_1 and assume that N is orientable and irreducible. Suppose that $\Phi \times I$ is a maximal product region for N and that $\Psi \times I$ is any product region in N. Then there is an ambient isotopy of N which takes $\Psi \times I$ into $\Phi \times I$.*

Proof. Suppose that there is a component, A, of $\partial_v \Psi \times I$ which is not contained in $\Phi \times I$. Then A is a vertical annulus. Because each component of Ψ contains a component of ∂S_0, we may choose an arc, γ, in Ψ from ∂S_0 to a point, x, on $A \cap S_0$. There is an path, δ, which runs along γ then runs round $A \cap S_0$ and then back along γ, and by moving this path slightly we may suppose δ is an arc embedded in Ψ. The vertical square $\delta \times I$ contained in $\Psi \times I$ may be isotoped rel vertical boundary into $\Phi \times I$. It is now easy to isotope the rest of A into $\Phi \times I$. We do this for each component of $\partial_v \Psi \times I$. After these isotopies $\partial_v \Psi \times I$ is contained in $\Phi \times I$ and hence $\Psi \times I$ is contained in $\Phi \times I$. □

Theorem 2.5. *Suppose that N is a relative cobordism and that N is orientable and irreducible. If $\Phi \times I$ and $\Psi \times I$ are maximal product regions of N, then there is an ambient isotopy of N which takes $\Psi \times I$ onto $\Phi \times I$.*

Proof. By Theorem 2.4 we can isotope $\Psi \times I$ into $\Phi \times I$ and we may also isotope $\Phi \times I$ into $\Psi \times I$. Combining these, we may isotope $\Psi \times I$ so that it is contained in $\Phi \times I$ and contains this manifold minus a collar. Uniqueness of collars now shows that $\Psi \times I$ may be isotoped to equal $\Phi \times I$. □

With the hypotheses of Theorem 1.4, define M^- to be the 3-manifold obtained from M by removing all boundary components of M which are disjoint from S. Define N to be the 3-manifold obtained by removing from M^- the interior of a regular neighborhood of S. We will regard the interior of N as equal to $M^- - S$. There are two copies, S_0 and S_1, of S in the boundary of N. Thus M^- is obtained from N by identifying S_0 with S_1 via a homeomorphism $\phi : S_1 \to S_0$. Since S is incompressible in M, it follows that N is a relative cobordism between S_0 and S_1. Let $\pi : \widetilde{M}^- \to M^-$ be the infinite cyclic cover dual to S, and let $\widetilde{S}_0$ be a lift of S to $\widetilde{M}^-$. Let τ be a generator of the group of covering transformations and set $\widetilde{S}_n = \tau^n \widetilde{S}_0$. We can regard $\widetilde{M}^-$ as $\bigcup_i N_i$ where N_i is a copy of N with $N_i \cap N_{i+1} = \widetilde{S}_i$ and $\tau N_i = N_{i+1}$. Define Y_n to be the submanifold $Y_n = \bigcup_{i=1}^n N_i$ of $\widetilde{M}^-$. Then Y_n is a relative cobordism between $\widetilde{S}_0$ and $\widetilde{S}_n$. Since M^- is irreducible it follows from the equivariant sphere theorem [12,4,9] that $\widetilde{M}^-$ is irreducible. Now Y_n is a submanifold of $\widetilde{M}^-$ bounded by incompressible surfaces hence Y_n is irreducible. Define P_n to be a maximal product region in Y_n.

Lemma 2.6. *With the hypotheses of Theorem* 1.4, *after an isotopy of P_k in Y_k we may arrange that for $i \leqslant k$ that $P_k \cap Y_i$ is contained in the product region P_i of Y_i.*

Proof. Suppose that P is a component of P_k. The first step is to arrange that for all $0 \leqslant i \leqslant k$, that $R = \widetilde{S}_i \cap P$ is a connected incompressible surface in P. This is already true for $i = 0, k$. Now ∂R is contained in $\partial_v P$ which is a union of annuli. Suppose there is a component, C, of ∂R which bounds a disc, D, in $\partial_v P$. Then, since $\widetilde{S}_i$ is incompressible, C bounds a disc, Δ, in $\widetilde{S}_i$. If we choose C innermost on $\widetilde{S}_i$ then $D \cup \Delta$ is a sphere which bounds a ball, B, in Y_k since Y_k is irreducible. Also the interior of B is disjoint from $\partial_v P$. Thus we can isotope D across the ball to remove C. We may thus assume that every component of ∂R is essential in $\partial_v P$.

Suppose that D is a compressing disc for R in P. Since $\widetilde{S}_i$ is incompressible in Y_k, there is a disc Δ in $\widetilde{S}_i$ with the same boundary as D. Then Δ intersects $\partial_v P$ in circles, and since $\partial_v P$ is incompressible in Y_k, these circles bound discs in $\partial_v P$. But these circles are in ∂R and so there are no such circles. Thus Δ is contained in P, hence $\Delta \subset R$ so R is incompressible in P.

Now $P \cong I \times \Phi$ for some compact connected surface Φ and R is an incompressible surface, possibly disconnected, in $I \times \Phi$ such that ∂R is contained in $I \times \partial\Phi$. Suppose that R_0 is a component of R. The projection of $I \times \Phi$ onto $0 \times \Phi$ maps R_0 π_1-injectively and sends boundary to boundary. Applying Theorem 13.1 of [6] to R_0 either this map is homotopic to a covering, or R_0 is an annulus and the map is homotopic rel boundary into $\partial\Phi$. In this case, both boundary components of R_0 are on the same annulus component of $I \times \partial\Phi$. Thus there is an annulus, B, in ∂P with the same boundary as R_0 and $B \cup R_0$ is the boundary of a solid torus in P. Then, after choosing an innermost such solid torus, we can isotope B across this solid torus to remove R_0 from R. Thus we may assume the map of $R_0 \to \Phi$ is homotopic to a covering. Define C to be a component of $\partial\widetilde{S}_0 \cap \Phi$. Recall that every component of a product region contains a component of $\partial\widetilde{S}_0$. In particular there is a component, C', of ∂R_0 which maps onto C. But C' is the only component of $\partial\widetilde{S}_i$ which maps to C and so $R_0 = R$ is connected.

Now P is a product $I \times \Phi$. Also R is an incompressible surface in P which separates P. In addition, R is disjoint from $\partial I \times \Phi$. It follows from the homotopy cobordism theorem for Haken manifolds that the submanifold of P between $0 \times \Phi$ and R is a product region for Y_n. Since Y_n is irreducible, Theorem 2.4 applied to Y_n implies that this product region may be isotoped into P_i. $\square$

It now follows from Lemma 2.6 that there is an ambient isotopy of Y_k taking $P_{k+1} \cap Y_k$ into P_k, thus we may assume that

$$Y_k \cap P_{k+p} \subset P_k.$$

Thus the compact subsurfaces $A_k = \widetilde{S}_0 \cap P_k$ are decreasing in other words $A_{k+1} \subset A_k$, and each is π_1-injective. We will define $A_0 = \widetilde{S}_0$.

Lemma 2.7. *With the hypotheses of Theorem* 1.4*, then for each* $k \geqslant 0$*, one of the following occurs*:

- A_k *is a regular neighborhood of* $\partial\widetilde{S}_0$ *in* $\widetilde{S}_0$.
- $\chi(A_{k+1}) < \chi(A_k)$.

Proof. Since all these subsurfaces are incompressible (and none are discs) it follows that $\chi(A_{k+1}) \leqslant \chi(A_k)$ with equality if and only if A_{k+1} is isotopic to A_k. Suppose that A_{k+1} is isotopic to A_k. Let

$$\tau : \widetilde{M}^- \to \widetilde{M}^-$$

be the generator of the group of covering transformations such that $\tau \widetilde{S}_0 = \widetilde{S}_1$. Then $P_{k+1} \cap \tau Y_k$ is a product region in τY_k and can therefore be isotoped in τY_k into the maximal such product region which is τP_k. Therefore the surface $R = P_{k+1} \cap \widetilde{S}_1$ can be isotoped in $\widetilde{S}_1$ into τA_k. Since P_{k+1} is a product, R is diffeomorphic to A_{k+1}, hence to A_k. Now R is a π_1-injective subsurface of τA_k, and R is diffeomorphic to τA_k. Therefore R can be isotoped in $\widetilde{S}_1$ so that $R = \tau A_k$. The submanifold $Q = P_{k+1} \cap Y_1$ now has boundary

$$\partial Q \cong A_k \cup (I \times \partial A_k) \cup \tau A_k.$$

The image, πQ, of Q in M^- is obtained from Q by identifying A_k and τA_k via τ. Thus πQ has boundary consisting of tori

$$\partial(\pi Q) \cong S^1 \times \partial A_k.$$

Since every essential torus in M is boundary parallel, $\partial(\pi Q)$ is contained in a regular neighborhood of ∂M, hence ∂A_k is contained in a regular neighborhood of $\partial \widetilde{S}_0$. Since M does not fiber over the circle with fiber S, the product region of N is not all of N so A_k is a proper subsurface of $\widetilde{S}_0$. Thus A_k is contained in a regular neighborhood of $\partial \widetilde{S}_0$. □

Lemma 2.8. *With the hypotheses of Theorem* 1.4, *for n sufficiently large every vertical square in Y_n is inessential.*

Proof. By Lemma 2.7, $\chi(A_k)$ is strictly decreasing. Also since A_k is an incompressible subsurface of $\widetilde{S}_0$ it follows that $\chi(\widetilde{S}_0) \leqslant \chi(A_k)$. Hence for sufficiently large n we have that A_n is a regular neighborhood of $\partial \widetilde{S}_0$. Hence the product region P_n is contained in a regular neighborhood of ∂Y_n. But now by Theorem 2.3 every vertical square in Y_n is inessential. □

The following argument was told to us by Gabai.

Lemma 2.9. *Suppose that M and N are irreducible relative cobordisms which do not contain essential vertical squares. Suppose that*

$$\partial_h M = S_0 \cup S_1, \qquad \partial_h N = T_0 \cup T_1$$

and that $\phi : S_1 \to T_0$ is a homeomorphism. Let P be the 3*-manifold formed from M and N by identifying S_1 with T_0 via ϕ. Then P has incompressible boundary.*

Proof. Suppose that D is a compressing disc for P. Note that P is a relative cobordism between S_0 and T_1. We may isotope D so that

- D is transverse to $S_1 \equiv T_0$.
- The intersection of ∂D with $\partial_v P$ consists of a vertical arcs.

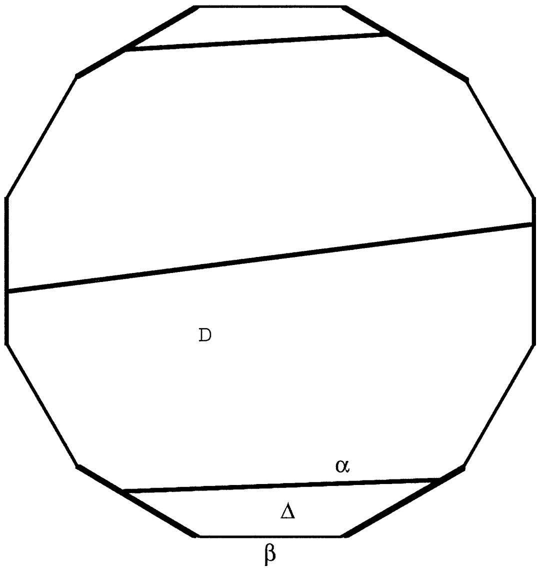

Fig. 1. Outermost arc on D.

Choose D, subject to the above, so that it has the minimum number of vertical arcs in its boundary. If there are no vertical arcs, then D is either contained in M or in N. Without loss, we suppose that D is contained in M. Then D is a compressing disc for S_0 in M, but the hypothesis that M is a relative cobordism includes that S_0 is incompressible. Hence we have a contradiction. The boundary of D is made up of horizontal and vertical arcs which alternate. This is shown in Fig. 1, where the vertical arcs are shown thicker.

Observe that since M and N are irreducible and they are glued together along incompressible surfaces, then P is also irreducible and $S_1 \equiv T_0$ is incompressible in P. The intersection of D with S_1 consists of arcs and circles. Since S_1 is incompressible in P, each such circle bounds a disc in S_1 and so we may isotope D to remove the circles. Thus each vertical side of D has an endpoint of exactly one arc of intersection. Now choose an outermost (on D) arc, α, of intersection on D with S_1. We claim that the situation is as shown in Fig. 1.

Thus there is a disc, Δ, in D with boundary the union of α and an arc β in ∂D and β intersects exactly two vertical arcs, γ, δ. Now Δ is contained in either M or in N, without loss of generality we will assume that Δ is contained in M. Thus Δ is a vertical disc in M and so Δ is boundary parallel. Hence D can be isotoped to reduce the number of vertical arcs in its boundary, a contradiction. □

Proof of Theorem 1.4. By Lemma 2.8 there is $n > 0$ such that the submanifold Y_n of $\widetilde{M}^-$ contains no essential vertical square. Now $Y_{2n} = Y_n \cup \tau^n Y_n$ and $Y_n \cap \tau^n Y_n = \widetilde{S}_n$ is incompressible. Thus we may apply Lemma 2.9 to Y_{2n} and deduce that Y_{2n} has incompressible boundary. Since S is incompressible in M it follows that ∂Y_{2n} is also incompressible in $\widetilde{M}^-$. Now there is an annulus, A, in $Y_{2n} \cap \partial \widetilde{M}^-$ with boundary consisting of two loops which are lifts of α. Let $F = \partial Y_{2n} - int(A)$ isotoped to a properly embedded surface in $\widetilde{M}^-$. Since ∂Y_{2n} is incompressible it follows that F is also incompressible and boundary-incompressible. Observe that F may be constructed in a finite cyclic cover. □

3. Virtually embedded slopes for the figure eight knot

In this section, M denotes the exterior of the figure eight knot K. This is a punctured torus bundle over the circle. We regard the punctured torus, T_0, as a square with a disc removed from the middle and opposite sides identified.

Let x, y be simple closed curves on T_0 given by the sides of the square. We regard the square as sitting in the xy-plane and then D_x denotes a right-handed Dehn-twist about the loop x, and D_y denotes a *left*-handed Dehn-twist about the loop y. The monodromy for the figure eight knot is $g = D_x \circ D_y$ (i.e., twist first about y then about x) thus

$$M \cong \frac{T_0 \times [0,1]}{(g(s),0) \sim (s,1)}.$$

We choose a base point, b, on the boundary of the puncture so that $\alpha = b \times [0,1]/\sim$ is a meridian for K, and $\beta = \partial T_0$ is a longitude for K. Now consider the bundle $M_f = T_0 \times I/f$ for $f = (D_x^{-1} \circ D_y^5)^2$. It was shown in [2, Lemma 7.2] that M_f double covers M with the meridian $\alpha_f = b_0 \times I/f$ of M_f projecting to the loop $\alpha^2\beta^{-1}$ in M.

Next consider the irregular 10-fold cover $F \to T_0$, shown in Fig. 3, to which f lifts. This cover is constructed by taking a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ cover of T_0, cutting along the two vertical arcs pictured, and then identifying the side labeled 1 (respectively 2) on one arc to the side labeled 1 (respectively 2) on the other arc.

The surface F has eight boundary components, 2 of which cover the boundary of T_0 degree-2; these boundary components are labeled 7 and 8. The other 6 boundary components are labeled 1–6, and cover the boundary of T_0 degree-1. Since both D_x and D_y^5 lift to F, $f = (D_x^{-1} \circ D_y^5)^2$ lifts as well. Indeed, D_y^5 lifts to simultaneous twists about $\tilde{y}_1$ and $\tilde{y}_2$ while $\widetilde{D}_x$ can be viewed as a 1/2 fractional twist about $\tilde{x}_5$ and $\tilde{x}_6$. If we require our lifts to fix the lifted basepoint, $\tilde{b}$, then $\widetilde{D}_x$ fixes pointwise rows 1 and 3 while shifting rows 2, 4 and 5 each one square to the right (mod 2). Let $\tilde{f}$ be the lift of f fixing $\tilde{b}$, and $\widetilde{M} = F \times I/\tilde{f}$. Then $\widetilde{M} \to M_f$ is a 10-fold cover (which gives a 20-fold cover of M).

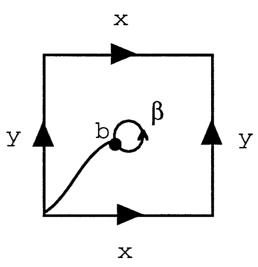

Fig. 2. A punctured torus.

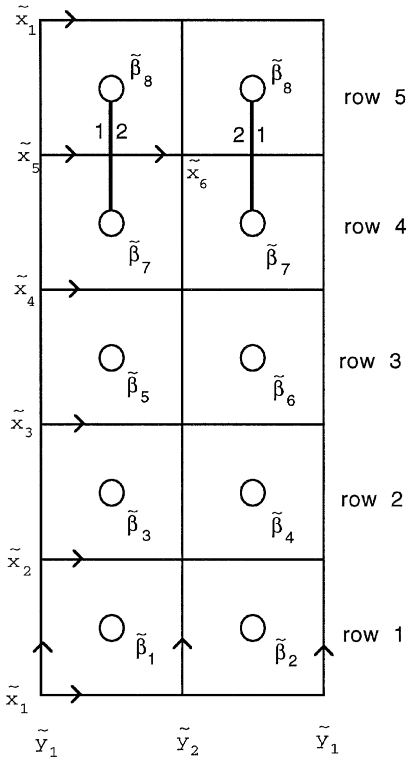

Fig. 3. Cross joins in F.

Note that $\widetilde{f}$ fixes the eight boundary circles of F. Denote by t_i a lift to the ith boundary component $\partial\widetilde{M}$ of the loop α_f on ∂M_f. Denote by $\widetilde{\beta}_i$ the ith boundary component of F which covers the boundary β of T_0 either degree-1 (for $i \leqslant 6$) or degree-2 (for $i = 7, 8$). Thus $(t_i, \widetilde{\beta}_i)$ is a homology basis for the ith boundary torus of $\widetilde{M}$. The loops t_i projects to $\alpha^2\beta^{-1}$ and $\widetilde{\beta}_i$ projects to β or β^2 in ∂M. We claim the following homology relations hold in $H_1(\widetilde{M})$

$$t_1 - t_2 - t_5 + t_6 = 0, \tag{1}$$

$$t_3 - t_1 + t_5 - t_7 + \widetilde{\beta}_3 + \widetilde{\beta}_4 + \widetilde{\beta}_5 + \widetilde{\beta}_6 = 0. \tag{2}$$

The first of these is in [2, Lemma 7.3]. The second one is derived below. Define *Ker* to be the kernel of the map induced by inclusion

$$incl_* : H_1(\partial\widetilde{M}) \to H_1(\widetilde{M}).$$

Let

$$p : H_1(\partial\widetilde{M}) \to H_1(\widetilde{T}_6)$$

denote the map given by the projection obtained from the direct sum decomposition of $H_1(\partial\widetilde{M})$ coming from $\partial\widetilde{M} = \widetilde{T}_6 \cup (\partial\widetilde{M} - \widetilde{T}_6)$ where $\widetilde{T}_6$ is the 6th boundary component.

Then $p(Ker) = H_1(\widetilde{T}_6)$, since the image of the left hand side of (1) is t_6 and the image of the left hand side of (2) is $\widetilde{\beta}_6$ which are a homology basis of $H_1(\widetilde{T}_6)$. Given z in *Ker*, there is a compact, oriented, 2-sided surface V properly embedded in $\widetilde{M}$ with boundary representing $[\partial V] = z$. Observe that no classes in boundary component number 8 appear in (1) and (2). It follows that, by adding discs and annuli to all boundary components of V on $\widetilde{T}_8$ that we may arrange that V is disjoint from $\widetilde{T}_8$. This implies that V is not the fiber of any fibration of $\widetilde{M}$. We may thus apply Theorem 1.4 to V and deduce that $p(z)$ is a virtual boundary slope of M_f. Since $p(z)$ is an arbitrary element of $H_1(\widetilde{T}_6)$ it follows that every slope of M_f is a virtual boundary slope. Since M_f is the 2-fold cover of the exterior of the figure 8 knot, this proves Theorem 1.3.

4. Calculations

In this section we derive relation (2).

We will derive the following relation in $H_1(\widetilde{M})$

$$t_3 - t_1 + t_5 - t_7 + x_2 - x_4 = 0 \tag{3}$$

then (as seen in Fig. 4) using that

$$x_2 - x_4 = \widetilde{\beta}_3 + \widetilde{\beta}_4 + \widetilde{\beta}_5 + \widetilde{\beta}_6$$

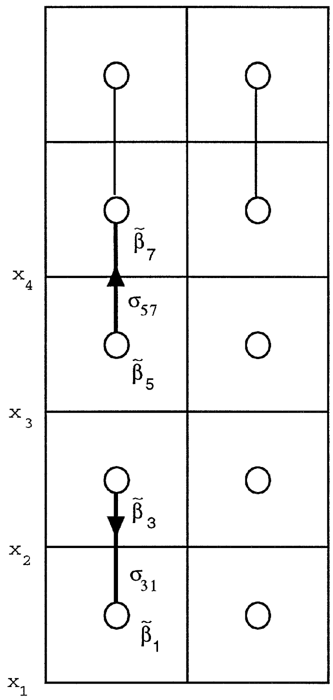

Fig. 4. Obtaining relation (2).

the relation (2) follows. One computes $t_3 - t_1$ as follows. Let σ_{31} be a simple path in $F \times \{0\}$ from $t_3 \cap F$ to $t_1 \cap F$ as shown in Fig. 4. Then the disc $\sigma_{31} \times I$ contained in $F \times I$ has boundary which gives the relation

$$t_3 - t_1 + \left(\widetilde{f}_*[\sigma_{31}] * \sigma_{31}^{-1}\right) = 0.$$

Here $*$ denotes composition of paths. Similarly, referring to Fig. 4, we obtain

$$t_5 - t_7 + \left(\widetilde{f}_*[\sigma_{57}] * \sigma_{57}^{-1}\right) = 0.$$

One then verifies that

$$\left(\widetilde{f}_*[\sigma_{31}] * \sigma_{31}^{-1}\right) + \left(\widetilde{f}_*[\sigma_{57}] * \sigma_{57}^{-1}\right) = x_2 - x_4.$$

5. Varying the slope for punctured torus bundles

We answer Question 1.7 for punctured torus bundles. Let $f : T_0 \to T_0$ be a homeomorphism of a punctured torus, and $M_f = T_0 \times I/f$ is a fibered 3-manifold with this monodromy. There is $k > 0$ such that $g \equiv f^k$ is congruent mod 2 to the identity in $SL_2\mathbb{Z}$. Then M_g is a k-fold cyclic cover of M_f. Consider the 4-fold $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover $\widetilde{T_0}$ of T_0. Then g is covered by a map $\widetilde{g}$ of T_0 which preserves boundary components. Also $N = M_{\widetilde{g}}$ is a 4-fold cover of M_g. Now N has 4 boundary components and they are all tori. On torus i, with $1 \leqslant i \leqslant 4$, there is a curve λ_i which covers ∂T_0. Now perform 0-Dehn filling on T_0 to obtain a torus bundle M_f^+. This is covered by a manifold N^+ obtained by doing Dehn-fillings on the four boundary components of N by attaching solid tori whose meridian discs cap off λ_i. Then N^+ covers M_f^+ and is therefore a torus bundle. Suppose that every $\lambda_i = 0$ in $H_1(N)$, then $\beta_2(N^+) \geqslant 4$. Since N^+ is a torus bundle over S^1 it follows that $\beta_2(N^+) \leqslant 3$ which gives a contradiction.

References

[1] M. Baker, On boundary slopes of immersed incompressible surfaces, Ann. Inst. Fourier (Grenoble) 46 (5) (1996) 1443–1449.
[2] M. Baker, On coverings of figure eight knot surgeries, Pacific J. Math. 150 (2) (1991) 215–228.
[3] D. Cooper, D.D. Long, Virtually Haken Dehn filling, Preprint, 1996.
[4] M.J. Dunwoody, An equivariant sphere theorem, Bull. London Math. Soc. 17 (5) (1985) 437–448.
[5] A. Hatcher, On the boundary curves of incompressible surfaces, Pacific J. Math. 99 (1982) 373–377.
[6] J. Hempel, 3-Manifolds, Ann. of Math. Stud. 86, Princeton Univ. Press, Princeton, NJ, 1976.
[7] J. Hempel, Coverings of Dehn fillings of surface bundles. II, Topology Appl. 26 (1987) 163–173.
[8] W. Jaco, Lectures on Three-Manifold Topology, CBMS Ser. 43, 1977.
[9] W. Jaco, J.H. Rubinstein, PL equivariant surgery and invariant decompositions of 3-manifolds, Adv. Math. 73 (2) (1989) 149–191.
[10] K. Johanson, Homotopy Equivalences of 3-Manifolds with Boundaries, Lecture Notes in Math., Vol. 761, Springer, Berlin, 1979.

[11] J. Maher, Virtually embedded boundary slopes, Topology Appl. 95 (1999) 63–74.
[12] Meeks, Simon, Yau, Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature, Ann. of Math. 116 (3) (1982) 621–659.
[13] U. Oertel, Boundaries of injective surfaces, Topology Appl. 78 (3) (1997) 215–234.