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ON THREE-MANIFOLDS WITH BOUNDED GEOMETRY

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ABSTRACT. In this note we combine some of Cheeger-Gromov's results [CG1,CG2,CG3] from the geometry of collapses of Riemannian 3-manifolds together with some three-dimensional topology to prove results which say that there are at most finitely many diffeomorphism classes of prime non-geometrizable three-manifolds which admit a metric of *bounded geometry* (i.e. with bounded sectional curvatures and bounded volume).

0. Introduction

Definition. A compact orientable 3-manifold is *geometrizable* if it has a splitting along a finite collection of disjoint essential spheres and tori into finitely many compact 3-manifolds whose interiors each admit a complete homogeneous riemannian metric (after capping off their boundary spheres by balls).

Thurston's geometrization conjecture states that all 3-manifolds are geometrizable.

There are eight homogeneous riemannian metric, which are locally modelled on the following 3-dimensional geometries: S^3, E^3, H^3, $S^2 \times E^1$, $H^2 \times E^1$, Nil, $\widetilde{SL_2(\mathbf{R})}$ and Sol.

A 3-manifold M is:

• **prime** if it is not the connect sum of two 3-manifolds neither of which is S^3.

• **irreducible** if every smoothly embedded sphere in M bounds a ball M.

• **∂-irreducible** if for every smooth properly embedded disc D in M there is a ball $B \subset M$ and a disc $D' \subset \partial M$ such that $\partial B = D \cup D'$.

• **atoroidal** if every $\mathbf{Z}^2$ subgroup in $\pi_1 M$ is conjugate into $\pi_1 \partial M$ and in addition $\pi_1 M$ does not contain the fundamental group of the klein bottle.

A prime orientable 3-manifold which is not irreducible is homeomorphic to $S^2 \times S^1$, and hence geometric. An irreducible orientable 3-manifold such that every $\mathbf{Z}^2$ subgroup of $\pi_1 M$ is conjugate into $\pi_1 \partial M$ is either atoroidal, or else the orientable I-bundle over the Klein bottle which is geometric.

By Thurston's hyperbolization theorem [Th2](cf. [Ka], [Ot1,2]) and the Torus theorem ([CJ],[Ga]), a non-geometrisable prime 3-manifold is irreducible, atoroidal and does not contain any embedded, incompressible, orientable surface. In particular it has an empty boundary.

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Given a positive real number $v > 0$, let $\mathcal{M}(v)$ be the set of diffeomorphism classes of closed orientable 3-manifolds which admit a Riemannian metric g with bounded sectional curvature $|K_g| \leq 1$ and bounded volume $vol(M, g) \leq v$.

There are infinitely many geometrizable 3-manifold in $\mathcal{M}(v)$. In fact by the work of Cheeger and Gromov [CG1,CG2] all closed graph 3-manifolds (to be defined in §1) belong to $\mathcal{M}(v)$ for any $v > 0$, and this characterizes graph 3-manifolds. More precisely, there is a constant $v_0 > 0$ such that: $\forall v \leq v_0, \mathcal{M}(v) = \mathcal{M}(v_0)$ is the set of closed graph 3-manifolds.

By the work of Jørgensen and Thurston, for v sufficiently large (eg. bigger or equal to the hyperbolic volume of the figure eight knot complement), there are infinitely many closed hyperbolic 3-manifolds in $\mathcal{M}(v)$.

The main result of this note is the following finiteness result concerning non-geometrizable prime summands of 3-manifolds in $\mathcal{M}(v)$.

Theorem (0.1). *Given $v > 0$ there is only a finite set $\mathcal{NG}(v)$ of orientable, non-geometrizable 3-manifolds that may occur as a prime summand in the connected sum decomposition of a 3-manifold in $\mathcal{M}(v)$. Moreover for every 3-manifold in $\mathcal{M}(v)$ the number of such non-geometrizable prime summands is bounded above by a number $p(v)$ depending only on v.*

As straightforward corollaries we obtain:

Corollary (0.2). *There is a constant $n(v)$ depending only on v such that $\mathcal{M}(v)$ contains at most $n(v)$ prime 3-manifolds which are not geometrizable.*

Corollary (0.3). *There is a constant $s(v)$ depending only on v such that $\mathcal{M}(v)$ contains at most $s(v)$ homotopy spheres.*

Definition. For a compact orientable 3-manifold M, let $\mathrm{Minvol}(M) = \inf\{vol(M, g)\}$ where g runs over all Riemannian metrics on $int(M)$ with bounded curvature $|K_g| \leq 1$.

Let $\mathcal{A}$ denote the set of compact orientable irreducible and atoroidal 3-manifolds, with zero Euler characteristic, and which do not admit a spherical metric (such a manifold is not a graph manifold). We denote by $\mathcal{H} \subset \mathcal{A}$ the subset of 3-manifolds which admit a complete hyperbolic structure of finite volume. By Thurston's hyperbolization theorem for Haken 3-manifolds, a manifold with non empty boundary in $\mathcal{A}$ belongs to $\mathcal{H}$. Thurston's geometrisation conjecture states that $\mathcal{H} = \mathcal{A}$.

When M admits a complete hyperbolic structure of finite volume g_0, a deep result, due to Besson-Courtois-Gallot [BCG] in the closed case and to Boland-Connel-Souto [BCS] in the cusp case, shows that the hyperbolic metric realizes the Minvol i.e. $\mathrm{Minvol}(M) = Vol(M, g_0)$.

Since there is no graph 3-manifold in $\mathcal{A}$, it follows from Cheeger-Gromov's work [CG1,CG2] that a 3-manifold in $\mathcal{A}$ has a strictly positive Minvol. By Corollary (0.2), for a given value $v > 0$, the set $\{\mathrm{Minvol} \leq v\} \cap \mathcal{A}$ contains at most finitely many prime, non-geometrizable 3-manifolds since they belong to $\mathcal{M}(v + 1)$.

Since the geometrizable 3-manifolds in $\mathcal{A}$ are exactly the subset $\mathcal{H}$ of hyperbolic 3-manifolds, the following result is a direct consequence of [BCG], [BCS], [Th,chap.5] and of Corollary (0.2). It shows that the set of values of the Minvol for manifolds in $\mathcal{A}$ behave like the set of volumes of hyperbolic manifolds.

Corollary (0.4). *The map* $\mathrm{Minvol} : \mathcal{A} \to (0, +\infty)$ *is finite to one and the set of values* $\mathrm{Minvol}(\mathcal{A} \backslash \mathcal{H})$ *is discrete. In particular the set of values* $\mathrm{Minvol}(\mathcal{A})$ *is a well-ordered subset of* $\mathbf{R}_+$ *whose limit points coincide with the limit points of the subset* $\mathrm{Minvol}(\mathcal{H})$*.*

There are two parts in the proof of Theorem (0.1). The first part (cf. §1) follows from Cheeger-Gromov's theory of collapses for riemannian manifolds with bounded sectional curvature. The second part (cf. §2) is a generalization of Thurston's hyperbolic Dehn filling theorem to the case of *graph-fillings*

1. Thick parts of Riemannian manifolds with bounded volume

A phenomenon which has received much attention in all dimensions from geometers is the notion of *collapse* : we say that a family of Riemannian metrics on a manifold collapses *with bounded geometry* if all the sectional curvatures remain bounded while the injectivity radius goes uniformly everywhere to zero.

For example any flat torus T^n collapses to any small dimensional torus T^k with $k < n$ by rescaling the metric on some of the S^1 factors.

Cheeger and Gromov [CG1,CG2] have proved that a necessary and sufficient condition for a manifold to have such a collapse with bounded geometry is the existence of a "generalized torus action" which they call an **F-structure**. F stands for "flat" in this terminology.

Intuitively an F-structure corresponds to different tori of varying dimension acting locally on finite coverings of open subsets of the manifold. Certain compatibility conditions on these local actions on intersections of these open subsets will insure that the manifold is partitioned into disjoint orbits of positive dimension. A precise definition of an F-structure can be given using the notion of sheaf of local groups actions, but we will not need it here.

A compact orientable 3-manifold M with an F-structure admits a partition into orbits which are circles and tori, such that each orbit has a saturated subset. A 3-manifold M has a *graph structure* in the sense of Waldhausen [Wa] and is a *graph manifold* if it can be obtained by glueing Seifert fiber spaces together along torus boundary components. These tori are not required to be incompressible. It follows from the definition of F-structure that such a partition corresponds to a *graph structure* on M (see [Ro1,§3]).

Another description of the family of all graph manifolds is that they are precisely those compact three manifolds which can be obtained, starting with the family of compact geometric *non-hyperbolic* three-manifolds, by the operations of connect sum and of glueing boundary tori together. Thus they arise naturally in both the Geometrization conjecture and in Riemannian geometry.

The aim of this section is to prove the following proposition which is true in any dimension:

Proposition (1.1). *Let M be a closed Riemannian n-manifold with $|K_g| \leq 1$ and $vol(M,g) \leq v$. Then M has a decomposition $M = N \cup G$ into two compact n-submanifolds such that:*
- *G admits an F-structure such that $\partial N = \partial G$ is an union of orbits.*
- *N belongs, up to diffeomorphism, to a finite set $\mathcal{N}(n,v)$ of smooth, compact, orientable n-manifolds.*

Here is a straightforward corollary in dimension 3:

Corollary (1.2). *Every manifold $M \in \mathcal{M}(v)$ has a decomposition $M = N \cup G$ into two compact (maybe not connected) 3-submanifolds such that:*
- *G is a (maybe empty) graph manifold.*
- *N belongs, up to homeomorphism, to a finite set $\mathcal{N}(v)$ of compact orientable 3-manifolds with zero Euler characteristic.* □

Riemannian geometry takes an important part in the proof of Proposition (1.1). This proposition is the analogue in bounded variable curvature of Jørgensen's finiteness theorem [Thm 5.12], which states that all complete hyperbolic 3-manifolds of bounded volume can be obtained by surgery on a finite number of cusped hyperbolic 3-manifolds. The finiteness of hyperbolic manifolds with volume bounded above and injectivity radius bounded below is a precursor to Gromov's compactness theorem, while the Margulis lemma takes the place of the Cheeger-gromov thick/thin decomposition [CG2, Thm.0.1].

The following theorem is a precise version of Cheeger-Gromov's thick/thin decomposition (see [CFG, Thm.1.3 and 1.7] for a proof). We recall that the ε-thin part of a Riemannian n-manifold (M,g) is the set of points $\mathcal{F}(\varepsilon) = \{x \in M\,, inj(x,g) < \varepsilon\}$

Theorem (1.3). *For each n, there is a constant μ_n, depending only on the dimension n, such that for any $0 < \varepsilon \leq \mu_n$ and any complete Riemannian n-manifold (M,g) with $|K_g| \leq 1$, there exists a Riemannian metric g_ε on M such that:*

(1) *The ε-thin part $\mathcal{F}(\varepsilon)$ of (M, g_ϵ) admits an F-structure compatible with the metric g_ε, whose orbits are all compact tori of dimension ≥ 1 and with diameter $< \varepsilon$.*

(2) *The Riemannian metric g_ε is ε-quasi-isometric to g and has bounded covariant derivatives of curvature, i.e. it verifies the following properties:*
- *$e^{-\varepsilon} g_\varepsilon \leq g \leq e^{\varepsilon} g_\varepsilon$.*
- *$\|\nabla^g - \nabla^{g_\varepsilon}\| \leq \varepsilon$, where ∇ and ∇^{g_ε} are the Levi-Civita connections of g and g_ε respectively.*
- *$\|(\nabla^{g_\varepsilon})^k R_{g_\varepsilon}\| \leq C(n,k,\varepsilon)$, where the constant C depends only on ε, the dimension n and the order of derivative k.*

Using Cheeger-Gromov's chopping theorem [CG3, Thm.0.1] one can prove the following:

Proposition (1.4). *For each integer $n \geq 2$, there are constants $\mu_n > 0$, $\Lambda_n > 0$, $\delta_n > 0$ and $c_n > 0$, depending only on n, such that for any closed Riemannian n-manifold (M, g) with $|K_g| \leq 1$, there is a metric g_n which is μ_n-quasi-isometric to g, with $|K_{g_n}| \leq \Lambda_n$ and a decomposition $M = N \cup G$ where:*

- *G is a compact n-submanifold which admits an F-structure compatible with g_n and $\partial N = \partial G$ is saturated.*
- *The injectivity radius for g_n at every point $x \in N$ verifies $inj(x, g_n) \geq \delta_n$.*
- *The second fundamental form of ∂N for the metric induced by g_n is bounded: $\|II^{g_n}_{\partial N}\| \leq c_n$.*
- *The volume $vol(\partial N, g_n) \leq c_n \cdot vol(M, g_n)$.*

Proof. We apply theorem (1.3) with the constant $\varepsilon = \mu_n$. So there is a metric g_n which is μ_n-quasi-isometric to g and such that $M = \mathcal{B}(\mu_n) \cup \mathcal{F}(\mu_n)$, where $\mathcal{B}(\mu_n) = \{x \in M\,, inj(x, g) \geq \mu_n\}$ and the μ_n-thin part $\mathcal{F}(\mu_n)$ admits an F-structure compatible with g_n. Moreover, since the covariant derivatives of the curvature of g_n have bounded norm by theorem (1.3), it follows that there is a constant $\Lambda_n > 0$ such that $|K_{g_n}| \leq \Lambda_n$. Therefore by the uniform decay of injectivity radius [GLP, Prop.8.22], there is a universal function $\phi_n(-,-)$, depending only on n, such that: $\forall x, x' \in M\,, inj(x', g_n) \geq \phi_n(inj(x, g_n), d_n(x, x'))$.

If $\mathcal{B}(\mu_n) = \emptyset$, we take $N = \emptyset$ and $G = M$.

We assume for the rest of the proof that $\mathcal{B}(\mu_n) \neq \emptyset$. We denote by d_n the distance on M associated with the metric g_n. Let $X \subset \mathcal{F}(\mu_n)$ be the set of points: $X = \{x \in \mathcal{F}(\mu_n)\,, d_n(x, \partial(\mathcal{B}(\mu_n)) \geq 1 + 2\mu_n\}$.

If $X = \emptyset$, then every point of M is at distance less than $2(1 + \mu_n)$ from a point of $\mathcal{B}(\mu_n)$. It follows from the uniform decay of injectivity radius that $inj(x, g_n) \geq \phi_n(\mu_n, 2(1 + \mu_n)) = \delta_n$ for every point $x \in M$. So we take $N = M$ and $G = \emptyset$.

If $X \neq \emptyset$, let $\mathcal{F}(X)$ be the union of all the orbits of points in X for the F-structure on $\mathcal{F}(\mu_n)$, compatible with g_n. It is a compact saturated subset of $\mathcal{F}(\mu_n)$. Since the diameter of the orbits of the F-structure is at most μ_n, it follows that $d_n(y, \partial(\mathcal{B}(\mu_n)) > 1$ for all point $y \in \mathcal{F}(X)$. In particular the closed tubular neighborhood of radius 1 around $\mathcal{F}(X)$, $T_1(\mathcal{F}(X))$, is contained in $\mathcal{F}(\mu_n)$. Since the local torus groups act by isometries, the equivariant form of Cheeger-Gromov's chopping theorem [CG3, Thm.0.1] (see also [Ro2, Thm.2.1]), shows that there is a compact n-submanifold $U \subset M$ with smooth boundary ∂U such that for some constant $c_n > 0$ depending only on n:

- $\mathcal{F}(X) \subset U \subset T_1(\mathcal{F}(X)) \subset \mathcal{F}(\mu_n)$ and U is saturated for the F-structure;
- $\|II^{g_n}_{\partial U}\| \leq c_n$;
- $vol(\partial U, g_n) \leq c_n \cdot vol(T_1(\mathcal{F}(X)), g_n) \leq c_n \cdot vol(M, g_n)$.

We set $G = U$ and $N = M \backslash \mathrm{int}(U)$. Since $X \subset U$, for every point $x \in N$ we have $d_n(x, \partial(\mathcal{B}(\mu_n)) \leq 2(1 + \mu_n)$. By the uniform decay of injectivity radius [GLP, Prop.8.22], we obtain as above that $inj(x, g_n) \geq \delta_n$ for every point $x \in N$. ☐

Proof of Proposition (1.1). By proposition (1.4), for some constants $\mu_n > 0$, $\Lambda_n > 0$, $\delta_n > 0$ and $c_n > 0$, depending only on n there is a metric g_n on M, which is μ_n-quasi-isometric to g, with $|K_{g_n}| \leq \Lambda_n$ and a decomposition $M = N \cup G$ such that:

• G is a compact n-submanifold which admits an F-structure compatible with g_n and $\partial N = \partial G$ is saturated.
• The injectivity radius for g_n at every point $x \in N$ verifies $inj(x, g_n) \geq \delta_n$.
• The second fundamental form of ∂N for the metric induced by g_n is bounded: $\|II^{g_n}_{\partial N}\| \leq c_n$.
• The volume $vol(\partial N, g_n) \leq c_n \cdot vol(M, g_n)$.

In particular, the volume of (M, g_n) verifies: $vol(M, g_n) \leq V(n, v)$ for a constant $V(n, v)$ depending only on μ_n and v, and thus only on n and v. Since $|K_{g_n}| \leq \Lambda_n$ and $inj(x, g_n) \geq \delta_n$ for every point $x \in N$, the diameter of N verifies: $diam(N, g_n) \leq D(n, v)$, where the constant $D(n, v)$ depends only on $v(n, v), \delta_n$ and Λ_n, and hence only on n and v.

To show that N belongs , up to diffeomorphism, to a finite set $\mathcal{N}(n, v)$ of smooth, compact, orientable n-manifolds, we use S. Kodani's extension [Ko] of Gromov's convergence theorem to some classes of Riemannian manifolds with boundary.

Let i_∂ be the infimum of inward normal injectivity radii of the boundary points of N. Then i_∂ is the infimum of the focal radius of ∂N and of half the length of a shortest geodesic which orthogonally intersects ∂N at the end points. (cf. [Ko, Lemma 6.3]). Let i_N be the minimum of i_∂ and the infimum of the injectivity radii of points at distance greater than i_∂ from ∂N. If $i_N < i_\partial$, then i_N is the infimum of the conjugate radii and of half the lengths of geodesic loops with base points at distance at least i_∂ from ∂N. In order to apply Kodani's results we need to have a lower bound on i_N, therefore we need to control the inward normal injectivity radius to ∂N. To do so the idea is to add a collar to ∂N. The following construction has been pointed out by J. Porti.

Since ∂N is a hypersurface in M, the uniform bounds $|K_{g_n}| \leq \Lambda_n$ and $\|II^{g_n}_{\partial N}\| \leq c_n$ imply that the focal radius of ∂N in M is bounded below by a constant $r_n = \frac{1}{\sqrt{\Lambda_n}} \arctan(\frac{\sqrt{\Lambda_n}}{c_n})$. Therefore the exponential map $exp : \nu^-_{\frac{r_n}{2}}(\partial N) \to M$ is a smooth immersion, where $\nu^-_{\frac{r_n}{2}}(\partial N)$ is the subspace of the normal bundle of ∂N which consists of normal vectors of length smaller or equal to $\frac{r_n}{2}$ and pointing ouside N. We use the exponential map to pull back the Riemannian metric g_n of M onto the collar $\nu^-_{\frac{r_n}{2}}(\partial N)$ of ∂N. We glue this collar to N along ∂N to get a Riemannian manifold N' with the same topological type as N and endowed with the metric g'_n which coincides with g_n on N and with the pull back metric on the collar $\nu^-_{\frac{r_n}{2}}(\partial N)$.

By [KO, Lemmas 3.1 and 3.2], see also [BZ, Chap. 6], the norm of the jacobian of the exponential map is uniformly bounded on $\nu^-_{\frac{r_n}{2}}(\partial N)$ above by a constant b_n and below by a constant $a_n > 0$, which depend only on Λ_n and c_n. It follows that the Riemannian metric (N', g'_n) has the following properties:
• $|K_{g'_n}| \leq \Lambda'_n$ where Λ'_n depends only on Λ_n, c_n, a_n, b_n, hence only on n.
• $\|II^{g'_n}_{\partial N'}\| \leq c'_n$, where the constant c'_n depends only on Λ_n, c_n, a_n, b_n by [KO, Lemma 3.1].
• $vol(N', g_n) \leq (1 + (a_n)^{-n}) vol(M, g_n) \leq (1 + (a_n)^{-n}) V(n, v) = V'(n, v)$.
• $i_{N'} \geq \delta'_n$, where δ'_n depends only on n, Λ'_n, c'_n, b_n and r_n, thus only on n . This follows from the uniform decay of injectivity radius in M, the uniform upper

bound on the jacobian of the exponential map and the uniform lower bounds on the conjugate radius of N' and focal radius of $\partial N'$.

• $diam(N', g'_n) \leq D'(n,v)$, since the volume of N' is bounded above by a constant $V'(n,v)$ and the injectivity radius of N' is bounded below by a constant δ'_n.

Therefore (N', g'_n) belongs to the class of n-dimensional compact Riemannian manifolds with bounded sectional curvature $|K_{g'_n}| \leq \Lambda'_n$ and a lower bound on the injectivity radius $i_{N'} \geq \delta'_n$. Moreover, if $\partial N' \neq \emptyset$, $\|II^{g'_n}_{\partial N'}\| \leq c'_n$. It follows from [GLP, Prop.7.5] and [Ko, Thm.A] in the case with boundary, that the Gromov-Hausdorff and the Lipschitz topology coincide for this class of manifolds. Furthermore $vol(N', g_n) \leq V'(n,v)$ and $diam(N', g_n) \leq D'(n,v)$, so the Riemannian manifold (N', g'_n) belongs to a class of riemannian manifolds which is precompact for the Gromov-Hausdorff topology by [GLP, Prop.5.2], and thus for the bilipschitz topology. It follows from the definition of the bilipschitz topology that there are, up to diffeomorphism, only finitely many manifolds in a precompact family with respect to this topology. Therefore there are, up to diffeomorphism, only finitely many manifolds N' and hence only finitely many manifolds N. □

2. Graph-fillings

Definition. A *graph-filling* of a compact orientable 3-manifold N is the operation of gluing a compact orientable (maybe not connected) graph 3-manifold G to N by identifying some toral components of ∂N with some toral components of ∂G.

A graph-filling is a generalization of a Dehn filling where each connected component of G is a solid torus.

Corollary (1.2) implies that every $M \in \mathcal{M}(v)$ either is a graph manifold, or belongs to $\mathcal{N}(v)$, or is obtained from a manifold in $\mathcal{N}(v)$ by a graph filling. Hence Theorem (0.1) is a straightforward consequence of Corollary (1.2) and the following result:

PROPOSITION (2.1). *Let M be a compact orientable 3-manifold with non empty boundary a collection of tori. There is only a finite set $\mathcal{NG}(M)$ of compact, orientable, non-geometrizable 3-manifolds that may occur as prime factors of the connected sum decompositions of all the compact, orientable 3-manifolds obtained by graph fillings of M. Moreover the number of such prime factors (counted with multiplicity) is also bounded above by a constant depending only on M.*

The purpose of this section is to prove Proposition (2.1). Before starting the proof we give some definitions.

Definition. Let M be a compact orientable 3-manifold and let $T \subset \partial M$ be a boundary torus. A *slope* $\alpha \in H_1(T, \mathbf{Z})$ is a homology class corresponding to an essential simple closed curve on T. We denote by $M(\alpha)$ the compact orientable 3-manifold obtained by Dehn filling T with slope α i.e. by gluing a solid torus

$S^1 \times D^2$ along T in such way that the boundary of a meridian disk $\{*\} \times \partial D^2$ has slope α on T. By convention ∞ will denote the empty slope, so $M(\infty)$ means that no Dehn filling occurred along T.

Definition. Let V be a soli torus, a *cable space* is the complement of an open tubular neighborhood of a (r, s)-cable of the core of V, where r, s are coprime integers with $s \geq 2$. It has a Seifert fibration over an annulus with one single cone point.

Definition. A compact orientable 3-manifold H is *hyperbolicabled* if there is a finite (maybe empty) set of disjoint compact cable subspaces $C_1, \ldots, C_k$ in H such that $C_i \cap \partial H$ is a torus component of ∂C_i, for $i = 1, \ldots, k$, and that $H_0 = H \backslash \cup_{i=1}^k C_i$ is not empty and admits a complete hyperbolic metric of finite volume on its interior. When the family of cable subspaces $\{C_i\}_{i=1,\ldots,k}$ is empty, the manifold H is said to be *hyperbolic*. Observe that a hyperbolicabled manifold is geometrizable.

The following lemma is a straightforward extension of Thurston's hyperbolic Dehn filling Theorem [Th1, Chap 5] :

Lemma (2.2). *Let H be a compact, orientable, hyperbolicabled 3-manifold , with q toral boundary components $T_1, \ldots, T_q$. Then on each torus component $T_i \subset \partial H$ there is a finite exceptional set of slopes $\mathcal{S}_i$ such that for any collection of slopes $(\alpha_1, \ldots, \alpha_q) \in (H_1(T_1, \mathbf{Z}) \cup \{\infty\} \backslash \mathcal{S}_1) \times \ldots (H_1(T_q, \mathbf{Z}) \cup \{\infty\} \backslash \mathcal{S}_q)$, the 3-manifold $H(\alpha_1, \ldots, \alpha_q)$ obtained by Dehn filling of H is irreducible, $\partial-$irreducible and geometrizable.*

Proof. Let $H_0 = H \backslash \cup_{i=1}^k C_i$ be the hyperbolic part of H, with $k \leq q$. By Thurston's hyperbolic Dehn filling theorem [Th1, Chap. 5], on each torus component $T'_i \subset \partial H_0$, $i = 1, \ldots, q$, there is a finite exceptional set of slopes $\mathcal{S}'_i$ such that for any collection of slopes $(\beta_1, \ldots, \beta_q) \in (H_1(T'_1, \mathbf{Z}) \cup \{\infty\} \backslash \mathcal{S}'_1) \times \ldots (H_1(T'_q, \mathbf{Z}) \cup \{\infty\} \backslash \mathcal{S}'_q)$, the 3-manifold $H_0(\beta_1, \ldots, \beta_q)$ obtained by Dehn filling of H_0 admits a complete hyperbolic structure of finite volume on its interior.

Let $T_i \subset \partial H$ be a boudary component. If $T_i = T'_i \subset \partial H_0$, then the exceptional set of slopes $\mathcal{S}_i = \mathcal{S}'_i$. Otherwise $T_i \subset \partial C_i$, where C_i is a cable subspace of H and $T'_i = \partial C_i \backslash T_i \subset H_0$.

If intersection number of the slope $\alpha \subset T_i$ with the fibre $f \subset T_i$ of the Seifert fibration of C_i is $|\Delta(\alpha, f)| \geq 2$, then the Dehn filled 3-manifold $C_i(\alpha)$ is a Seifert manifold over a disk, with two exceptional fibres and incompressible boundary. Hence gluing $C_i(\alpha)$ to a boundary component of an hyperbolic 3-manifold still yields an irreducible, $\partial-$irreducible and geometrizable 3-manifold.

If $|\Delta(\alpha, f)| = 1$, then $C_i(\alpha)$ is a solid torus. A homological calculation shows that the intersection numbers of two slopes β and β' on T'_i corresponding to the boundaries of meridian disks of $C_i(\alpha)$ and $C_i(\alpha')$ verifies: $|\Delta(\beta, \beta')| = s_i^2 |\Delta(\alpha, \alpha')|$, where $s_i \geq 2$ is the order of the exceptional fibre of C_i (cf. [Go, Lemma 3.3]). Then the existence of a finite exceptional set of slopes $\mathcal{S}'_i$ on $T'_i \subset \partial H_0$ implies the existence of a finite exceptional set of slopes $\mathcal{S}_i$ on T_i. $\square$

Let M be a compact irreducible and ∂-irreducible, orientable 3-manifold with non-empty boundary a finite collection of tori. Using the *JSJ-decomposition* it is easy to show that M contains a finite (possibly empty) minimal collection $\mathcal{T}$ of disjoint essential tori such that the closure of each component of $M \backslash \mathcal{T}$ is either a graph or a hyperbolicabled 3-manifold each of whose cable subspaces contains a boundary component of M. It is a subcollection of the *JSJ-family* of tori of M. One calls $\mathcal{T}$ the *reduced JSJ-family* of tori.

Let $T \subset \partial M$ be a torus component and let W_T be the closure of the connected component of $M \backslash \mathcal{T}$ containing T in its boundary.

Definition. A *bad* slope $\alpha \subset T$ is a slope such that either:
• W_T is a graph manifold and $W_T(\alpha)$ is either reducible, or ∂-compressible, or
• W_T is hyperbolicabled and α belongs to the exceptional set of slopes $\mathcal{S} \subset T$ given by the lemma (2.2).

The following is a generalization of the previous lemma (2.2).

LEMMA (2.3). *Let M be a compact, connected, orientable, irreducible and ∂ irreducible 3-manifold with non-empty boundary a finite collection of tori. Suppose also that M is not a cable space. Then on each torus component $T \subset \partial M$ there are only finitely many bad slopes.*

Proof. Let $\mathcal{T} \subset M$ be the reduced JSJ-family of tori and let W_T be the closure of the connected component of $M \backslash \mathcal{T}$ containing T.

We claim that W_T is not a cable-space. To see this, suppose that W_T is a cable space. Then $\partial W_T = T \cup T'$. If $T' \subset \partial M$ then since M is connected we have $M = W_T$, which contradicts our hypothesis. Otherwise T' is also a boundary component of some other component, C, of the reduced JSJ decomposition. By definition of reduced JSJ decomposition we see that C is not hyperbolic. Thus C is a graph manifold. But then $C \cup W_T$ is also a graph manifold which contradicts the minimallity of the collection $\mathcal{T}$ of tori in the reduced JSJ decomposition. This proves the claim. Thus if W_T is a graph manifold it is not a cable space hence by [CGLS,§2] there are only finitely many bad slopes on T.

Otherwise, when W_T is hyperbolicabled the set of bad slopes on T is finite by Lemma (2.2). □

Proof of Proposition (2.1). Every graph filling of a graph manifold is a graph manifold and hence has a geometric decomposition. Thus if M is a graph manifold the set $\mathcal{N}G(M)$ is empty. Hence we may assume that M is not a graph manifold. By considering the connected sum decomposition of M in prime factors, one reduces the proof of Theorem (2.1) to the case where M is irreducible and not a graph manifold. In particular M is not a solid torus and is ∂-irreducible.

Since any connected sum factor of a graph manifold is a graph manifold, we have only to consider graph fillings by irreducible graph manifolds. Moreover M is geometrizable because it is irreducible and $\partial M \neq \emptyset$, hence graph fillings by irreducible and ∂-irreducible, orientable graph manifolds always yield geometrizable 3-manifolds. Therefore we have only to deal with Dehn fillings by solid tori, because an orientable, irreducible 3-manifold with a compressible torus in its boundary is a solid torus.

Now we argue by induction on the number of boundary components of M.

If there is only one boundary component since M is irreducible and ∂-irreducible, Lemma (2.3) shows that except for finitely many bad slopes $\alpha \subset \partial M$ the Dehn filled 3-manifold $M(\alpha)$ is irreducible and geometrizable. This proves Theorem (2.1) in this case.

Let $T_1, \dots, T_q$ be the boundary components of ∂M. By Lemma (2.3), except for a finite set of bad slopes $\mathcal{S}_i \subset T_i$ on each boundary torus, any collection of slopes $(\alpha_1, \dots, \alpha_q) \in (H_1(T_1, \mathbf{Z}) \cup \{\infty\} \backslash \mathcal{S}_1) \times \dots (H_1(T_q, \mathbf{Z}) \cup \{\infty\} \backslash \mathcal{S}_q)$, yields an irreducible and ∂-irreducible 3-manifold $M(\alpha_1, \dots, \alpha_q)$ which is geometrizable.

For any bad slope $\beta_i \in \mathcal{S}_i \subset T_i$, the Dehn filled manifold $M(\beta_i) = M(\infty, \dots, \alpha_i, \dots, \infty)$ is compact orientable with strictly less boundary tori than M. From the discussion above, clearly $\mathcal{NG}(M) \subset \cup \mathcal{NG}(M(\beta_i))$, where the union is taken over the finite set of all bad slopes in $\cup_{i=1}^q \mathcal{S}_i$. Then $\mathcal{NG}(M)$ is finite since by the induction hypothesis the sets $\mathcal{NG}(M(\beta_i))$ are finite. In the same way the number of non-geometrizable prime factors for any graph filling of M is bounded above by the maximum of non-geometrizable prime factors for the graph fillings of the manifolds $M(\beta_i)$ where β_i runs over all bad slopes in $\cup_{i=1}^q \mathcal{S}_i$. □

We can now prove the main theorem (0.1). By (1.2) there is a finite set $\mathcal{N}(v)$ of compact orientable 3-manifolds such that every $M \in \mathcal{M}(v)$ can be decomposed as $M = N \cup G$ with $N \in \mathcal{N}(v)$ and G a graph manifold. Then by (2.1) the set $\mathcal{NG}(N)$ is finite for each $N \in \mathcal{N}(v)$. The union of these finite sets as N varies over the finite set $\mathcal{N}(v)$ is $\mathcal{NG}(v)$ and is therefore finite. Furthermore the number of non-geometrizable prime summands is bounded by the maximum of the number of such summands that appear for any graph filling of any $N \in \mathcal{N}(v)$. Thus this bound, $p(v)$, depends only on the volume bound v. □

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