Virtually Haken fillings and semi-bundles

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Suppose that $M$ is a fibered three-manifold whose fiber is a surface of positive genus with one boundary component. Assume that $M$ is not a semi-bundle. We show that infinitely many fillings of $M$ along $\partial M$ are virtually Haken. It follows that infinitely many Dehn-surgeries of any non-trivial knot in the three-sphere are virtually Haken.

57M10; 57M25

1 Introduction

In this paper manifold will always mean a compact, connected, orientable, possibly bounded, three-manifold. A bundle means a manifold which fibers over the circle. A semi-bundle is a manifold which is the union of two twisted $I$–bundles (over connected surfaces) whose intersection is the corresponding $I$–bundle. An irreducible, $\partial$–irreducible manifold that contains a properly embedded incompressible surface is called Haken. A manifold is virtually Haken if has a finite cover that is Haken.

Waldhausen’s virtually Haken conjecture is that every irreducible closed manifold with infinite fundamental group is virtually Haken. It was shown by Cooper and Long [1] that most Dehn-fillings of an atoroidal Haken manifold with torus boundary are virtually Haken provided the manifold is not a bundle.

Theorem 1 Suppose that $M$ is a bundle with fiber a compact surface $F$ and that $F$ has exactly one boundary component. Also suppose that $M$ is not a semi-bundle and not $S^1 \times D^2$. Then infinitely many Dehn-fillings of $M$ along $\partial M$ are virtually Haken.

Corollary 2 Let $k$ be a knot in a homology three-sphere $N$. Suppose that $N - k$ is irreducible and that $k$ does not bound a disk in $N$. Then infinitely many Dehn-surgeries along $k$ are virtually Haken.

The main idea is to construct a surface of invariant slope (see Section 3) in a particular finite cover of $M$. Such surfaces are studied in arbitrary covers using representation theory in a sequel [2]. While writing this paper we noticed that Thurston’s theory
of bundles extends to semi-bundles, and in particular there are manifolds which are semi-bundles in infinitely many ways. We discuss this in the next section.

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## 2 Bundles and semi-bundles

Various authors have studied semi-bundles, in particular Hempel and Jaco [6] and Zulli [10; 11]. Suppose a manifold has a regular cover which is a surface bundle. We wish to know when a particular fibration in the cover corresponds to a bundle or semi-bundle structure on the quotient. The following has the same flavor as some results of Hass [5].

**Theorem 3** Let $M$ be a compact, connected, orientable, irreducible three-manifold, $p: \tilde{M} \to M$ a finite regular cover, and $G$ the group of covering automorphisms. Suppose that $\phi: \tilde{M} \to S^1$ is a fibration of $\tilde{M}$ over the circle. Suppose that the cyclic subgroup $V$ of $H^1(\tilde{M}; \mathbb{Z})$ generated by $[\phi]$ is invariant under the action of $G$. Then one of the following occurs:

1. The action of $G$ on $V$ is trivial. Then $M$ also fibers over the circle. Moreover there is a fibering of $M$ which is covered by a fibering of $\tilde{M}$ that is isotopic to the original fibering.

2. The action of $G$ on $V$ is non-trivial. Then $M$ is a semi-bundle. Moreover there is a semi-fibering of $M$ which is covered by a fibering of $\tilde{M}$ that is isotopic to the original fibering.

**Proof** Define $N = \ker[\phi_*: \pi_1 \tilde{M} \to \pi_1 S^1]$. Since $\phi$ is a fibration $N$ is finitely generated. If $N$ is cyclic then the fiber is a disc or annulus. In these cases the result is easy. Thus we may assume $N$ is not cyclic. Because $V$ is $G$–invariant, it follows that $N$ is a normal subgroup of $\pi_1 M$ and $Q = \pi_1 M/N$ is infinite. Using [6, Theorem 3] it follows that $M$ is a bundle or semi-bundle (depending on case 1 or 2) with fiber a compact surface $F$ and $N$ has finite index in $\pi_1 F$. The pull-back of this (semi)fibration of $M$ gives a fibration of $\tilde{M}$ in the cohomology class of $\phi$ and is therefore isotopic to the given fibration. \hfill $\square$

Suppose that $G \cong (\mathbb{Z}_2)^n$ acts on a real vector space $V$ and let $X = \text{Hom}(G, \mathbb{C})$ denote the set of characters on $G$. Then $X \cong \text{Hom}(G, \mathbb{Z}_2)$. For each $\epsilon \in X$ there is a $G$–invariant generalized $\epsilon$–eigenspace

$$V_{\epsilon} = \{ v \in V : \forall g \in G \ g \cdot v = \epsilon(g)v \}.$$
Then $V$ is the direct sum of these subspaces $V_\epsilon$.

Suppose that $M$ is an atoroidal irreducible manifold with boundary consisting of incompressible tori. According to Thurston there is a finite collection (possibly empty), $\mathcal{C} = \{C_1, \ldots, C_k\}$, called fibered faces. Each fibered face is the interior of a certain top-dimensional face of the unit ball of the Thurston norm on $H_2(M, \partial M; \mathbb{R})$. It is an open convex set with the property that fibrations of $M$ correspond to rational points in the projectivized space $\mathbb{P}(\cup_i C_i) \subset \mathbb{P}(H_2(M, \partial M; \mathbb{R}))$.

Let $G = H_1(M; \mathbb{Z}/2)$. The regular cover $\tilde{M}_s$ of $M$ with covering group $G$ is called the $\mathbb{Z}_2$–universal cover. Let $\mathcal{D} = \{D_1, \ldots, D_l\}$ be the fibered faces for this cover. For each $\epsilon \in H^1(M; \mathbb{Z}_2)$ there is an $\epsilon$–eigenspace $H_{2,\epsilon}$ of $H_2(\tilde{M}_s, \partial \tilde{M}_s; \mathbb{R})$. For each $1 \leq i \leq l$ and $\epsilon \in H^1(M; \mathbb{Z}_2)$ we call $S_{i,\epsilon} = D_i \cap H_{2,\epsilon}$ a semi-fibered face if it is not empty. It is the interior of a compact convex polyhedron whose interior is in the interior of some fibered face for $\tilde{M}_s$. Let $S_i$ be the union of the $S_{i,\epsilon}$ where $\epsilon$ is non-trivial.

**Theorem 4** With the above notation there is a bijection between isotopy classes of semi-fiberings of $M$ and rational points in $\mathbb{P}(\cup_i S_i)$.

**Proof** A semi-fibration of $M$ gives such a rational point by considering the induced fibration on $\tilde{M}_s$. The converse follows from Theorem 3. We leave it as an exercise to check uniqueness up to isotopy. □

We believe that all points in $\mathbb{P}(\cup_i S_i)$ correspond to isotopy classes of non-transversally-orientable, transversally-measured, product-covered 2–dimensional foliations of $M$. This is true for rational points and therefore holds on a dense open set (using the fact that the set of non-degenerate twisted 1–forms is open). However, since we have no use for this fact, we have not tried very hard to prove it.

**Definition** A manifold is a *sesqui-bundle* if it is both a bundle and a semi-bundle.

An example is the torus bundle $M$ with monodromy $-\text{Id}$. This is the quotient of Euclidean three-space by the group $G_2$ (Wolf [8, Theorem 3.5.5]). $M$ has infinitely many semi-fibrations with generic fiber a torus and two Klein-bottle fibers. In addition, $M$ is a bundle thus a sesqui-bundle.

A hyperbolic example may be obtained from $M$ as follows. Let $C$ be a 1–submanifold in $M$ which is a small $C^1$–perturbation of a finite set of disjoint, immersed, closed geodesics in $M$ chosen so that:

1. No two components of $C$ cobound an annulus and no component bounds a Mobius strip.
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intersects every flat torus and flat Klein bottle.

Each component of $C$ is transverse to both a chosen fibration and semi-fibration.

Let $N$ be $M$ with a regular neighborhood of $C$ removed. Then the interior of $N$ admits a complete hyperbolic metric. By (3) it is a sesqui-bundle. This answers a question of Zulli who asked in [11] if there are non-Seifert 3–manifolds which are sesqui-bundles.

### 3 Virtually Haken fillings

The following is well-known, but we include it here for ease of reference.

**Lemma 5** Suppose $M$ is Seifert fibered and has one boundary component. Then one of the following holds:

1. $M$ is $D^2 \times S^1$ or a twisted $I$–bundle over the Klein bottle.
2. Infinitely many Dehn-fillings are virtually Haken.

**Proof** The base orbifold $Q$ has one boundary component and no corners. If $\chi^{orb} Q > 0$ then $Q$ is a disc with at most one cone point thus $M = D^2 \times S^1$. If $\chi^{orb} Q = 0$ then $Q$ is a Mobius band or a disc with two cone points labeled 2 and in either case $Q$ has a 2–fold orbifold-cover that is an annulus $A$. But then $M$ is 2–fold covered by a circle bundle over $A$. Since $M$ is orientable it follows that this bundle is $S^1 \times A$ and hence $M$ is a twisted $I$–bundle over the Klein bottle.

Finally, if $\chi^{orb}(Q) < 0$ then all but one filling of $M$ is Seifert fibered. There are infinitely many fillings of $M$ which give a Seifert fibered space, $P$, with base orbifold $Q'$ and $\chi^{orb}(Q') < 0$. There is an orbifold-covering of $Q'$ which is a closed surface of negative Euler characteristic. The induced covering of $P$ contains an essential vertical torus and is therefore virtually Haken.

**Definitions** A slope on a torus $T$ is the isotopy class of an essential simple closed curve on $T$. We say that a slope lifts to a covering of $T$ if it is represented by a loop which lifts. The following is immediate:

**Lemma 6** Suppose $\mathcal{T} \to T$ is a finite covering. Then the following are equivalent:

1. Some slope on $T$ lifts to $\mathcal{T}$.
2. The covering is finite cyclic.
(3) Infinitely many slopes on \( T \) lift to \( \widetilde{T} \).

The distance, \( \Delta(\alpha, \beta) \), between slopes \( \alpha, \beta \) on \( T \) is the minimum number of intersection points between representative loops. If \( \alpha \) is a slope on a torus boundary component of \( M \) then \( M(\alpha) \) denotes the manifold obtained by Dehn-filling \( M \) using \( \alpha \). A surface \( S \) in a manifold \( M \) is essential if it is compact, connected, orientable, incompressible, properly-embedded, and not boundary-parallel. Let \( M \) be a manifold with boundary a torus and \( \alpha \subset \partial M \) a slope. Suppose that \( N \) is a finite cover of \( M \). An essential surface \( S \subset N \) has invariant slope \( \alpha \) if \( \partial S \neq \emptyset \) and every component of \( \partial S \) projects to a loop homotopic to a non-zero multiple of \( \alpha \). We call a finite cover \( p: N \to M \) a \( \partial \)–cover if there is an integer \( d > 0 \) and a homomorphism \( \theta: \pi_1(\partial M) \to \mathbb{Z}_d \) such that for every boundary component \( T \) of \( N \) we have \( p_*(\pi_1 T) = \ker \theta \). The existence of \( \theta \) ensures each component of \( \partial N \) is the same cyclic cover of \( \partial M \).

The following lemma reduces the proof of the main theorem to constructing an essential non-fiber surface of invariant slope in a \( \partial \)–cover of \( M \).

**Lemma 7** Suppose that \( M \) is a compact, connected, orientable irreducible 3–manifold with one torus boundary component. Suppose that there is a \( \partial \)–cover \( N \) of \( M \) and an essential non-separating surface \( S \subset N \) of invariant slope. Assume that \( S \) is not a fiber of a fibration of \( N \). Then \( M \) has infinitely many virtually-Haken Dehn-fillings.

**Proof** We first remark that the particular case that concerns us in this paper is that \( M \) is a bundle with boundary and thus \( M \) is irreducible. Since \( M \) is irreducible at most 3 fillings give reducible manifolds (Gordon and Luecke [4]). A cover of an irreducible manifold is irreducible (Meeks and Yau [7]). Therefore it suffices to show there are infinitely many fillings of \( M \) which have a finite cover containing an essential surface.

If \( M \) contains an essential torus then this torus remains incompressible for infinitely many Dehn-fillings by Culler–Gordon–Luecke–Shalen [3, Theorem 2.4.2]. If \( M \) is Seifert fibered then by Lemma 5 either the result holds or \( M = S^1 \times D^2 \) or is a twisted \( I \)–bundle over the Klein bottle. The latter two possibilities do not contain a surface \( S \) as in the hypotheses. By Thurston’s hyperbolization theorem we are reduced to case that \( M \) is hyperbolic.

Since \( p: N \to M \) is a \( \partial \)–cover there is \( d > 0 \) such that every component of \( \partial N \) is a \( d \)–fold cover of \( \partial M \). Let \( k \) be a positive integer coprime to \( d \). Let \( p_k: \widetilde{N}_k \to N \) be the \( k \)–fold cyclic cover dual to \( S \). We claim that there is a homomorphism \( \theta_k: \pi_1 M \to \mathbb{Z}_{kd} \) such that every slope in \( \ker \theta_k \) lifts to every component of \( \partial \widetilde{N}_k \).

Assuming this, the filling \( M(\gamma) \) of \( M \) is covered by a filling, \( \widetilde{N}_k(\gamma) \), of \( \widetilde{N}_k \) if and only if the slope \( \gamma \subset \partial M \) lifts to each component of \( \partial \widetilde{N}_k \). Since \( S \) is non-separating,
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Proof of Theorem 1.

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Proof of Theorem 1.

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We claim that there is an essential surface $S$.

Using real coefficients, all cohomology groups have direct-sum decomposition into $\pm 1$ eigenspaces for $\tau^*$.
with obvious notation, it swaps \( \mu_1 \) with \( \mu_2 \) and \( \lambda_1 \) with \( \lambda_2 \). If \( \epsilon = \pm 1 \) then \( V_\epsilon \) has basis \( \{ \mu_1 + \epsilon \mu_2, \lambda_1 + \epsilon \lambda_2 \} \) and thus has dimension 2. Let

\[
K = \text{Im} \left[ \text{incl}^*: H^1(\overline{W}; \mathbb{R}) \to H^1(\partial \overline{W}; \mathbb{R}) \right].
\]

Decompose \( K = K_+ \oplus K_- \). We claim that \( \dim(K_+) = \dim(K_-) = 1 \). Since \( \dim(K) = 2 \) the only other possibilities are that \( K_+ = V_+ \) or \( K_- = V_- \). The intersection pairing on \( \partial \overline{W} \) is dual to the pairing on \( H^1(\partial \overline{W}, \mathbb{R}) \) given by \( \langle \phi, \psi \rangle = (\phi \cup \psi) \cap [\partial \overline{W}] \). This pairing vanishes on \( K \). Since \( \langle \mu_1 + \epsilon \mu_2, \lambda_1 + \epsilon \lambda_2 \rangle = 2 \langle \mu_1, \lambda_1 \rangle = \pm 2 \), the restriction of \( \langle , \rangle \) to each of \( V_\pm \) is non-degenerate. This contradicts \( K = V_\pm \).

Choose a primitive class \( \phi \in H^1(\overline{W}; \mathbb{Z}) \) with \( \text{incl}^* \phi \in K_- \). Let \( S \) be an essential oriented surface in \( \overline{W} \) representing the class Poincaré dual to \( \phi \). Then \( \tau_*[S] = -[S] \) as required.

The 1–manifold \( \alpha_i = T_i \cap \partial S \) with the induced orientation is a 1–cycle in \( \partial \overline{W} \). Then \( [\partial S] = [\alpha_1] + [\alpha_2] \in H_1(\partial \overline{W}) \). Since \( T_i \) is a torus all the components of \( \alpha_i \) are parallel. Since \( \tau(T_i) = T_2 \) all components of \( \partial S \) project to isotopic loops in \( \partial W \) thus \( S \) has invariant slope for the cover \( \overline{W} \to M \). This gives:

**Case (i)** If \( S \) is not the fiber of a fibration of \( \overline{W} \) then the result follows from Lemma 7.

Thus we are left with the case that \( S \) is the fiber of a fibration of \( \overline{W} \). Let \( N \) be the \( \mathbb{Z}_2 \)–universal covering of \( W \). This is a regular covering and each component of \( \partial N \) is a two-fold cover of \( \partial W \). We claim that the composition of coverings \( N \to W \to M \) is regular.

Recall that a subgroup \( H < G \) is characteristic if it is preserved by \( \text{Aut}(G) \). The \( \mathbb{Z}_2 \)–universal covering \( N \to W \) corresponds to the characteristic subgroup \( \pi_1 N < \pi_1 W \). The cover \( W \to M \) is cyclic and so \( \pi_1 W \) is normal in \( \pi_1 M \). A characteristic subgroup of a normal subgroup is normal. Hence \( \pi_1 N \) is also normal in \( \pi_1 M \). This proves the claim. It follows that \( N \to M \) is a \( \partial \)–cover. A pre-image, \( \tilde{S} \), of \( S \) in \( N \) is a fiber of a fibration.

**Case (ii)** Suppose the one-dimensional vector space of \( H_2(N, \partial N; \mathbb{R}) \) spanned by \( [\tilde{S}] \) is invariant under the group of covering transformations of \( N \to M \).

Then, by Theorem 3, \( M \) is semi-fibered which contradicts our hypothesis. This completes case (ii). Therefore there is some covering transformation, \( \sigma \), such that \( \sigma_*[\tilde{S}] \neq \pm [\tilde{S}] \).
Because $\widetilde{S}$ and $\sigma \widetilde{S}$ are fibers, they both meet every boundary component of $N$. Since $S$ has invariant slope for the cover $N \to M$ it follows that $\widetilde{S}$ and $\sigma \widetilde{S}$ have the same invariant slope for this cover.

Case (iii) Suppose $S$ is a fiber and $[\partial \widetilde{S}] \neq \pm \sigma_* [\partial \widetilde{S}] \in H_1(\partial N)$.

Given a boundary component of $N$, there are integers $a$ and $b$ such that the class $a[\widetilde{S}] + b \cdot \sigma_* [\widetilde{S}] \in H_2(N, \partial N)$ is non-zero and represented by an essential surface $G$ that misses this boundary component. Thus $G$ is not a fiber of a fibration. Clearly $G$ has invariant slope. The result now follows from Lemma 7 applied to the surface $G$ in the $\partial$–cover $N$. This completes case (iii). The remaining case is:

Case (iv) $S$ is a fiber and there is $\epsilon \in \{ \pm 1 \}$ with $\sigma_* [\partial \widetilde{S}] = \epsilon \cdot [\partial \widetilde{S}] \in H_1(\partial N)$.

Consideration of the homology exact sequence for the pair $(N, \partial N)$ shows $x = \sigma_* [\widetilde{S}] - \epsilon \cdot [\widetilde{S}] \in H_2(N, \partial N)$ is the image of some $y \in H_2(N)$. Using exactness of the sequence again it follows that $y + \iota_* H_2(\partial N)$ is not zero in $H_2(N)/\iota_* H_2(\partial N)$. Hence every filling of $N$ produces a closed manifold with $\beta_2 > 0$. Infinitely many slopes on $\partial M$ lift to slopes on $\partial N$. The result follows. This completes the proof of case (iv) and thus of the Theorem 1.

Proof of Corollary 2 Let $\eta(K)$ be an open tubular neighborhood of $k$. By hypothesis the knot exterior $M = N \setminus \eta(K)$ is irreducible. Every semibundle contains two disjoint compact surfaces whose union is non-separating, thus the first Betti number with mod-2 coefficients of a semi-bundle is at least 2. Because $N$ is a homology sphere $H_1(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$, therefore $M$ is not a semi-bundle. Since $N$ is a homology sphere it, and therefore $M$, are orientable.

If $M$ is a bundle with fiber $F$ then, since $N$ is a homology sphere, $F$ has exactly one boundary component. Since $k$ does not bound a disk in $N$ it follows that $M \neq D^2 \times S^1$. The result now follows from Theorem 1. If $M$ contains a closed essential surface then infinitely many fillings are Haken, [3, Theorem 2.4.2]. The remaining possibilities are that $M$ is hyperbolic and not a bundle, or else Seifert fibered. The hyperbolic non-bundle case follows from [1].

This leaves the case that $M$ is Seifert fibered. The manifold $M$ is not a twisted $I$–bundle over the Klein bottle because the latter has mod-2 Betti number 2. The result now follows from Lemma 5.

References


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