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TOPOLOGY AROUND THE POINCARÉ CONJECTURE

DARYL COOPER

UCSB

This text approximates a transcript of a talk presented to the 50th annual meeting of the South African Mathematical Society on 1'st November, 2007. The talk was a survey, for a wide mathematical audience, of some of the mathematics inspired by the generalized Poincaré conjecture. None of the results are due to the author, and many key ideas have been left out. There is no attempt to mention all those who made significant contributions. In places we have over-simplified and omitted important ideas. The style of the talk was rather informal. I thank the organizers of the SAMS, in particular David Gay, for the opportunity to speak, and the mathematics department of the University of Cape Town for their hospitality during my visit, during which time this report was prepared

An n-dimensional *manifold* is a topological space that is locally like Euclidean n-space (second countable, Hausdorff, and every point has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$). A 2-manifold is called a *surface.* These are classified. The compact orientable ones are completely determined by the an integer called the *Euler characteristic,* and are the sphere, torus, 2-holed torus, 3-holed torus,...

An important n-manifold is the n-*sphere*

$$\begin{aligned} S^n &= \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \; : \; x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\,\} \\ &\cong \mathbb{R}^n \cup \{\infty\} \end{aligned}$$

Roughly speaking, the Generalized Poincaré conjecture (GPC) is:
A compact manifold *is equivalent* to S^n $\Leftrightarrow$ it has the same *algebraic topology invariants* as S^n.

To make this precise we need to discuss the *italicized* terms. A key concept of algebraic topology is the *fundamental group* of a topological space. Given a topological space X we would like to assign a group, $\pi_1(X)$, called the *fundamental group* of X based at p to X. An irritating fact for the beginner is that in order to define the group operation we need to choose a point p in X called the *base point.* However the isomorphism type of the fundamental group only depends on the path component of X that contains the basepoint. In particular, for a connected manifold this group is well defined up to isomorphism. This is why we avoid the traditional notation $\pi_1(X,p)$ here. A *loop* in X based at p is a continuos map $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = p$. Elements of $\pi_1(X)$ are *homotopy classes* (defined below) of loop starting and ending at p. For example a torus, T, is the product of two circles $S^1 \times S^1$ and

$$\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}.$$

The fundamental group of the torus is generated by the two loops $\alpha(t) = (exp(2\pi it), 1)$ and $\beta(t) = (1, exp(2\pi it))$ using the basepoint $p = (1,1) \in S^1 \times S^1$.

If α, β are two loops based at p, the group operation is given by $[\alpha] \cdot [\beta] = [\gamma]$ is the homotopy class of the loop γ which first goes round α then goes round β :

$$\gamma(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2. \\ \beta(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

It is at this juncture that one needs the loop α to end where loop β starts, and this why we need a base point. This product is usually not commutative, although it is for the torus. A space is *simply connected* if it is path connected and the fundamental group is *trivial* i.e. contains only one element. Another way to say this is that *every loop in M can be continuously shrunk in M to a point.* The original Poincaré conjecture (although asked rather than conjectured by Poincaré) is: If M is a compact, simply connected 3-manifold without boundary then M is the 3-sphere.

The idea of *homotopy* is a key idea in many parts of mathematics. It is a formalization of the idea on *continuously changing* one thing into another. Formally a *homotopy* is just a continuous map

$$H : X \times [0,1] \to Y.$$

For $t \in [0,1]$ the *time t map* is $H_t : X \to Y$ defined by $H_t(x) = H(t,x)$. One says that the 1-parameter family of maps $H_t : X \to Y$ *varies continuously* and describes the map H_1 as a *continuous deformation* of the map H_0. Another viewpoint is that the map $t \mapsto H_t$ is a continuous map of the interval into $Maps[X,Y]$ suitably topologized.

Two maps $f, g : X \to Y$ are *homotopic* written $f \simeq g$ if there is a homotopy H as above with $f = H_0$ and $g = H_1$. It is easy to check that $\simeq$ is an equivalence relation.

Example 1. Every map $f : X \to \mathbb{R}^n$ is homotopic to a constant map using $H(x,t) = t \cdot f(x)$.

Example 2. The unit circle is $S^1 \subset \mathbb{C}$. For each integer n define a map $f_n : S^1 \to S^1$ by $f_n(z) = z^n$. Then $f_n \simeq f_m \Leftrightarrow m = n$. The integer n is called the *degree* of the map and is a *homotopy invariant.*

We now come to the key idea of *homotopy equivalence;* algebraic topology is the study of spaces up to this equivalence relation. A *homotopy inverse* of a map $f : X \to Y$ is a map $g : Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. Incidentally in topology *map* means *continuous.* If one replaces $\simeq$ by $=$ then this is the definition of *homeomorphism.* A map with a homotopy inverse is called a *homotopy equivalence.* One says that X and Y are *homotopy equivalent* spaces, or they have the *same homotopy type,* if there is a homotopy equivalence $f : X \to Y$. This gives an equivalence relation between topological spaces. It turns out that many important properties of manifolds only depend on their homotopy type. The distinction between *geometric* and *algebraic* topology is the distinction between equivalence up to homeomorphism versus up to homotopy equivalence.

Example 1. $\mathbb{R}^n \simeq \{$*one point* $\}$

Example 2. $S^1 \times [0,1] \simeq S^1$.

Example 3. $GL(n,\mathbb{R}) \simeq O(n)$. The homotopy is given by the Gram-Schmidt orthonormalization process. Here $O(n)$ is the *orthogonal group* of $n \times n$ matrices A with $A^{-1} = A^t$.

A space X is called *contractible* if the identity map $id : X \to X$ is homotopic to a constant map. Another way to say this is to say X is contractible if it is homotopy equivalent to a point. Example (1) implies Euclidean space is contractible. It is interesting that Poincaré did not ask whether every contractible 3-manifold is $\mathbb{R}^3$, since this is perhaps the simplest 3-manifold. We will discuss the surprising answer to this question below.

The definition of fundamental group involves the notion of homotopy classes of loop. Two loops α, β base at p are *homotopic rel endpoints* and are *in the same homotopy class* if there is a homotopy H from α to β such that each time-t-map H_t is a loop starting and ending at p. In symbols this means $\forall\ t \quad H(0,t) = H(1,t) = p$, also $\alpha = H_0$ and $\beta = H_1$.

When $n \geq 4$ every finitely presented group is the fundamental group of some manifold of dimension n. The isomorphism problem for groups is logically undecidable, hence there is no algorithm to decide, in general, whether two compact 4-manifolds are homeomorphic or not. The same holds true in each dimension $n \geq 4$. In fact there is no algorithm to decide whether or not a group, given in terms of generators and relations between those generators, is the *trivial group* (contains exactly one element) or not. Thus one can not in general decide if a compact n-manifold with $n \geq 4$ is simply connected or not. However a consequence of Perelman's proof of Thurston's Geometrization conjecture is that there is a (non-conceptual) classification of compact orientable 3-manifolds, and an algorithm for deciding whether two 3-manifolds are homeomorphic or not.

Up to now we have avoided the issue of what it means to say two n-manifolds are *equivalent*. There are various answers, and one of the major triumphs of twentieth century topology was the understanding the differences between the various notions. Two manifolds X, Y *equivalent* if:

Answer 1. (algebraic topology answer) They are *homotopy equivalent.*

Answer 2. (point-set topology answer) They are *homeomorphic* i.e. there is a continuous bijection $h : X \to Y$ such that h^{-1} is also continuous.

Answer 3. (differentiable-topology answer) They are *diffeomorphic* i.e. the homeomorphisms h and h^{-1} are differentiable (or *smooth*).

Clearly $3 \Rightarrow 2 \Rightarrow 1$.

Generalized Poincaré Conjecture (GPC)
If a compact manifold is homotopy equivalent to S^n then M is *equivalent* to S^n.

The term "*equivalent*" might be taken to mean homeomorphic or diffeomorphic. A big surprise was that the answer to these questions turned out to be different, depending on the dimension n. As we shall see, in some sense the *right question* for GPC is (2), the point-set topology question.

An equivalent formulation of the GPC involves the *higher homotopy groups.* Given a space X and basepoint $p \in X$ the *n'th homotopy group of X based at p* is $\pi_n(X)$. An element is a homotopy class of map $f : S^n \to X$ (sending the basepoint of S^n to p). For $n \geq 2$ this group is abelian.

Intuitively $\pi_n(X) = 0$ iff every S^n in X can be shrunk in X to a point. If $k < n$ one may assume $f(S^k)$ misses the point $\infty \in S^n = \mathbb{R}^n \cup \{\infty\}$ and so $f(S^k) \subset \mathbb{R}^n$ and can therefore be shrunk there; hence $\pi_k(S^n) = 0$. It can be shown that $\pi_n(S^n) \cong \mathbb{Z}$. The issue of whether or not a given map is a homotopy equivalence is reduced to algebra using:

Theorem 1 (Whitehead's theorem). *If X and Y are connected manifolds then $f : X \to Y$ is a homotopy equivalence $\Leftrightarrow$ $\forall\ n > 0\ \ f_n : \pi_n(X) \to \pi_n(Y)$ is an isomorphism of groups. (Here f_n is the homomorphism of groups induced by f)*

In particular, using a bit more algebraic topology, one can reformulate the hypothesis of the GPC in various ways:

Theorem 2. *The following are equivalent for a compact, connected n-manifold M without boundary*
(1) M is homotopy equivalent to a S^n
(2) Every map of S^k into M with $k < n$ can be shrunk to a point in M.
(3) $\forall\ \ 1 \leq i < n\ \ \ \pi_i(M) \cong 0$
(4) M is simply connected and the homology groups $H_i(M) \cong 0$ vanish for $2 \leq i < n$.

Unfortunately, in dimension $n \geq 4$ it is not logically possible to always decide if a given n-manifold is simply connected. However Perelman's proof of the Geometrization Conjecture in dimension 3 implies that for 3-manifolds this is decidable.

In each odd dimension $n \geq 3$ there are n-manifolds called *Lens spaces* which are homotopy equivalent but not homeomorphic, thus the algebraic topology classification is different to the point-set topology classification. These manifolds are the quotient of the n-sphere, S^n, by a finite cyclic subgroup of $O(n+1)$ thought of as the group of isometries of S^n with its standard round metric. The following result was a big surprise. It gives a counterexample to the differentiable version of GPC.

Theorem 3 (Milnor). *There is a 7-dimensional manifold which is homeomorphic but not diffeomorphic to S^7.*

There is a beautiful theory of the possible exotic smooth structures on a given manifold, except in dimension 4 where there are still mysteries. Brieskhorn showed that the 28 differentiable structures on the

7-sphere are the intersection with the unit sphere in $\mathbb{C}^5$ of the algebraic hypersurfaces

$$z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0 \qquad 1 \le k \le 28.$$

Thus there is a link between the topology of singular points on algebraic varieties and topology. The only dimension in which it is unknown whether or not there is more than one smooth structure on S^n is $n = 4$ and this is the only unsolved case of the differentiable version of the GPC. In dimension $n < 7$ if $n \neq 4$ then a manifold is diffeomorphic to S^n iff it is homeomorphic to S^n. However the topological version of GPC is true in all dimensions (though the three proofs differ dramatically):

Theorem 4 (Smale (1960) for $n \ge 5$, Freedman (1982) for $n = 4$, and Perelman (2003) for $n = 3$). *If a smooth n-manifold is homotopy equivalent to S^n then it is homeomorphic to S^n.*

In dimension 3 two manifolds are homeomorphic iff they are diffeomorphic. As mentioned, the theory of *non-compact* 3-manifolds holds more surprises.

Theorem 5 (Whitehead). *There is a non-compact 3-manifold W (the* **Whitehead manifold***) such that $W \neq \mathbb{R}^3$ and $W \times \mathbb{R} = \mathbb{R}^4$. Hence $W \simeq \mathbb{R}^3$.*

The Whitehead manifold (actually there are uncountable many different ones) is constructed as an increasing union

$$W = \bigcup_{n=1}^{\infty} V_n \qquad V_1 \subset V_2 \subset V_3 \cdots .$$

Here each $V_i \cong S^1 \times \mathbb{R}^2 \equiv V$ is a *open solid torus*. Any finite union is homeomorphic to V, it is just an open solid torus and is not simply connected. However the infinite union is magical. It is reminiscent of the fact that

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \left(\frac{1}{n!}\right) \mathbb{Z}$$

exhibits the rationals $(\mathbb{Q}, +)$ as an additive abelian group which is an increasing union of infinite cyclic subgroups. Indeed there is a variant of Whitehead's construction which gives a 3-manifold whose fundamental

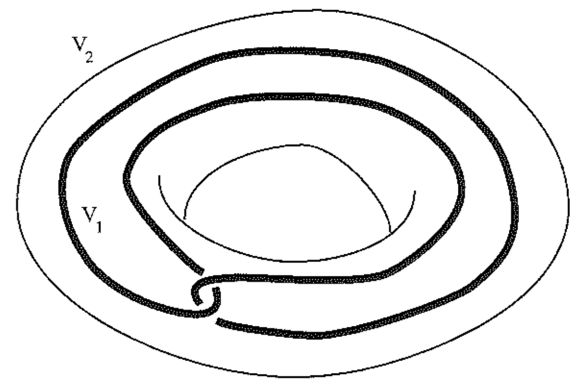


Figure 1. Constructing the Whitehead Manifold.

group is isomorphic to $(\mathbb{Q}, +)$ and which has universal cover $\mathbb{R}^3$. The key point in the construction of W is the way V_i sits inside V_{i+1}. This is "always the same" in the sense that each one is given by an injective map (topological embedding)

$$f : V_1 \to V_2$$

which has image $f(V_1)$ contained in a small neighborhood of a circle C in V_2 that is knotted.

The key point is that C can be shrunk to a point within V_2 but not without *crossing itself.* This is what is meant by saying that C is *knotted* in V_2. This ensures every loop in V_i can be shrunk to a point in V_{i+1}, implying W is simply connected. The proof that $W \neq \mathbb{R}^3$ involves a bit of topology. But the proof that $W \times \mathbb{R} = \mathbb{R}^4$ is just based on the observation that in 4 dimensions one *has enough room* to be able to undo knots. The extra dimension allows one to move C without crossing itself to C' where C' is a small unknotted circle in V_2. It is then easy to see from this that $W \times \mathbb{R} = \mathbb{R}^4$.

The remainder of the talk is an attempt to give some idea of the proof of the GPC for high dimensions $n \ge 5$. In particular we wish to give some idea of how the hypothesis is used, and why the conclusion is a homeomorphism not a diffeomorphism. This entails a brief discussion of a *handle decomposition* of a manifold. The relation of handles to homology groups. The idea of cancelling pairs of *complimentary handles* and the fact that the hypothesis implies the homology groups vanish which allows one to cancel all the handles except for just two. We do not have time to indicate how the dimension hypothesis $n \geq 5$ is needed to do moves which enable cancellation. However the discussion of the Whitehead manifold illustrates that extra dimensions sometimes allow geometric moves which give simplifications that are not possible in lower dimensions. First we discuss the *Alexander trick* that shows why a manifold which has a handle decomposition with only two handles is homeomorphic (but perhaps not diffeomorphic) to a sphere.

The *n-ball* is

$$D^n = \{ x \in \mathbb{R}^n : ||x|| \leq 1 \}.$$

It is a compact n-manifold with *boundary* an $(n-1)$-sphere:

$$\partial D^n = \{ x \in \mathbb{R}^n : ||x|| = 1 \} \cong S^{n-1}.$$

Observation. $S^n = D^n_+ \cup D^n_-$ is the union of two manifolds with boundary, each diffeomorphic to n-balls: the *upper hemisphere,* D^n_+ and the *lower hemisphere,* D^n_-. They intersect along the equatorial $(n-1)$-sphere

$$D^n_+ \cap D^n_- = \partial D^n_+ = \partial D^n_- = S^{n-1}.$$

Thus S^n is the result of gluing two n-balls, $A = D^n_+$ and $B = D^n_-$ together using a *particular* homeomorphism $h : \partial A \to \partial B$ to identify ∂A with ∂B. A key point is that *every choice of homeomorphism h gives a manifold that is homeomorphic, to (but perhaps not diffeomorphic to) a sphere.* This follows from

Theorem 6 (Alexander trick). *Every homeomorphism $h : \partial D^n \to \partial D^n$ extends to a homeomorphism $H : D^n \to D^n$ with $H|\partial D^n = h$.*

Proof.

$$H(x) = \begin{cases} 0 & if \quad x = 0 \\ ||x|| \cdot h(x/||x||) & if \quad x \neq 0 \end{cases} \qquad \square$$

Observe that H is usually not differentiable at 0.

The strategy to prove GPC in dimensions $n \geq 5$ is the following. Given $M^n \simeq S^n$ decompose $M = A \cup B$. Show A and B are n-balls with

$A \cap B = \partial A = \partial B$. The result follows from the above. In a bit more detail:

- Every manifold has a *handle decomposition.* Each *handle* is topologically an n-ball, and they are assembled in a special way.

- Use algebraic topology (homology theory) to *cancel handles* two-at-a-time until only two remain.

Smale's proof of the GPC used a *Morse function.* This is a differentiable function $f : M \to \mathbb{R}$ with the property that every critical point is *non-degenerate,* in the sense that the Hessian matrix is non-singular. The finitely many critical points give rise to a *handle decomposition* of M. The signature of the Hessian at the critical point is the *index* of the handle. However, one can work directly with handle decompositions without mentioning Morse functions, which is what we describe next.

We will now work with n-dimensional manifolds. Given $0 \leq k \leq n$ a *k-handle* or *handle of index* k is $h^k = D^k \times D^{n-k}$ and $D^k \times 0$ is called the *core* of the handle and D^{n-k} is called the *co-core.* One thinks of a k-handle as a k-cell, namely it's core, but it has been *fattened up* to be an n-dimensional manifold by taking the product with the co-core. Usually in manifold topology the superscript indicates the *dimension* of the manifold. But for a handle it indicates the index.

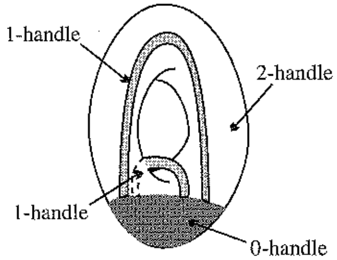
Suppose M is a manifold with boundary and $N = M \cup h^k$ where $h^k = D^k \times D^{n-k}$ is a k-handle and

$$M \cap h^k = (\partial D^k) \times D^{n-k} \subset \partial M.$$

We say that N is obtained from M by *attaching a k-handle.* The subset of $(\partial D^k) \times D^{n-k} \subset \partial M$ is called the *attaching region* of the handle h^k. From the point of view of homotopy theory the fattening of D^k, by taking the product with D^{n-k}, has no effect, so that the homotopy type of N is the same as if only the core $D^k \times 0 \subset h^k$ is added onto M. In this sense adding a k-handle to M is like adding a k-dimensional ball to M by attaching it along its boundary.

A *handle decomposition* of a compact manifold M is a sequence of submanifolds

$$M_0 \subset M_1 \subset M_2 \cdots \subset M_k = M$$



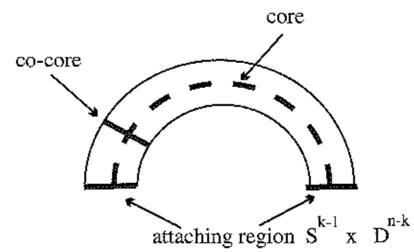


Figure 2. A handle decomposition of the torus.

A k-handle $h^k = D^k \times D^{n-k}$.

with M_0 an n-ball and M_{i+i} is obtained from M_i by attaching a handle.

The left side of figure 2 shows a decomposition of a torus into 4 handles: one 0-handle, two 1-handles, and one 2-handle. It is easy to show that every compact differentiable manifold has a finite handle decomposition. Furthermore, one can easily arrange that the handles are attached in order of increasing index, and that there is only one 0-handle only one n-handle. Thus

$$N_0 \subset N_1 \subset N_2 \cdots \subset N_n = M$$

with N_0 an n-ball thought of as one 0-handle. Furthermore

$$closure(N_i \setminus N_{i-1}) = \bigcup_{j=1}^{m_i} h_j^i$$

is the disjoint union of all the handles, H_j^i, of index i.

A *cancelling pair of handles* consists of two handles: h^k attached to M, and h^{k+1} attached to $M \cup h^k$, so that the *attaching sphere*, $(\partial D^{k+1}) \times D^{n-(k+1)}$, of h^{k+1} intersects the *belt sphere* $D^k \times \partial D^{n-k}$ transversally in exactly one point. It is then easy to see that $(M \cup h^k) \cup h^{k+1}$ is homeomorphic to M.

We now describe the process of *sliding one handle over another handle.* If h_1 and h_2 are disjoint handles of the same index k attached to M then $M \cup h_1 \cup h_2$ is diffeomorphic to $M \cup h_1 \cup h_3$ where h_3 is a k-handle with attaching sphere S_3^{k-1} which is obtained by tubing the attaching spheres S_1^{k-1} of h_1 and S_2^{k-1} of h_2 together using a thin tube (homeomorphic

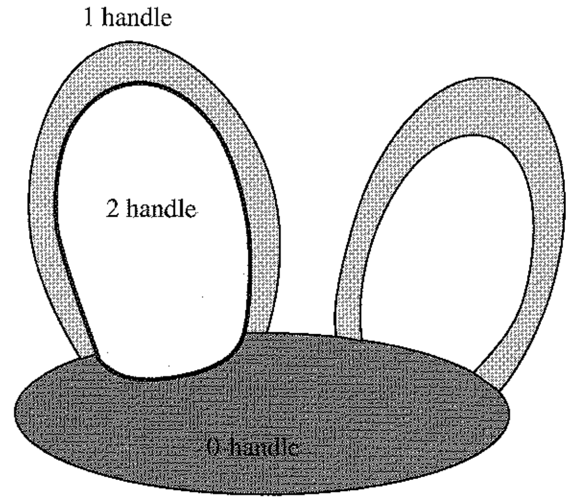


Figure 3. A 2-handle that cancels a 1-handle.

to $[0,1] \times S^{k-2}$) in a small neighborhood of some chosen arc $\alpha \subset \partial M$ connecting $S_1^{k_1}$ to S_2^{k-1}. We shall see below that the algebraic process of doing row operations on a matrix can be translated into a sequence of handle slides. The end result of these slides is that pairs of handles can be cancelled.

Given a manifold M and an integer $i \geq 0$ the *i'th homology group* $H_i(M)$ is an abelian group which depends only on the homotopy type of M. The definition of homology for arbitrary topological spaces is more complicated than the definition of the homotopy groups, but it turns out to be easier to calculate. For a manifold, this group has one generator for each handle of index i. It has one relation between these generators for each handle of index $i+1$. Of course it is far from clear with this description why the homology groups of M are independent of the handle decomposition of M that is used. It follows from this description, and the fact that S^n has a handle decomposition with just one handle of index 0 and one of index n, that when $n \geq 2$ that $H_i(S^n)$ is 0 unless $i \in \{0, n\}$ in which case it is $\mathbb{Z}$. Since the homology groups only depend on the *homotopy type* it follows that $M \simeq S^n$ then M and S^n have isomorphic homology.

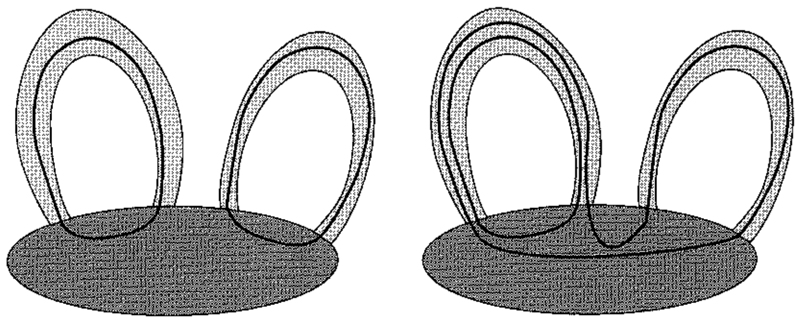


Figure 4. Sliding the attaching sphere of one handle over another handle corresponds to adding one row of the matrix A to another.

Given a finitely generated abelian group, G, (written additively) choose elements $g_1, \cdots g_n$ which generate G. Suppose that these elements satisfy the m relations

$$\sum_{j=1}^{n} A_{i,j} g_j = 0 \qquad 1 \le i \le m$$

where $A_{i,j}$ are certain integers. Furthermore suppose that every relation among these generators is a $\mathbb{Z}$-linear combination of these. Then A is called a *presentation matrix* for G. Each row of A corresponds to a relation between the generators. Column operations on A correspond to changing the generating set, and row operations correspond to changing the relations, but all these operations preserve the isomorphism type of G. Conversely an integral matrix determines a finitely generated abelian group up to isomorphism.

A handle presentation of M determines a presentation matrix (A_{ij}), of $H_p(M)$ as follows. The j'th column of the matrix A corresponds to the generator corresponding to a handle h_j^p of index p in N_p. The i'th row of the matrix A corresponds to the relation corresponding to a handle h_i^{p+i} of index $(p+1)$ in N_{p+1}. Then $A_{i,j}$ is the number of times the attaching sphere $S^p \times D^{p-1} \subset \partial h_i^{p+1}$ runs across h_j^p algebraically. More formally, the i'th row of A is given by the kernel of the map induced by inclusion $H_p(N_p) \to H_p(N_p \cup h_i^{p+1})$.

The hypothesis of the GPC that $M \simeq S^n$ implies $H_i(M) = H_i(S^n) = 0$ for $0 < i < n$. The fact that $H_i(M) = 0$ means that the presentation matrix, A, for $H_i(M)$ can be transformed by row operations (each of which adds or subtracts one row to/from another row) into the *obvious* presentation matrix, B, for the trivial group: the one which in which the i'th relation sets the i'th generator equal to zero, possibly followed by some redundant relations saying $0 = 0$.

$$A = \begin{pmatrix} 3 & -2 & \cdots \\ 1 & 4 & \cdots \\ \cdot & \cdot & \\ \cdot & \cdot & \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdots \\ \cdot & \cdot & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = B$$

Provided there are enough dimensions (i.e. $n \ge 5$) these row operations correspond to handle slides. The $\pm$ sign in the row operation corresponds to orientations of the two handles involved in the slide. The end result is a new handle presentation given by the simplified matrix B. From this one sees that each handle of index i is cancelled by the corresponding handle of index $(i+1)$. Thus we can cancel all the handles of index i. Continuing in this fashion one cancels all the handles except for two handles, one of index 0 and the other of index n. From this one deduces that M is homeomorphic to S^n.

This sketch has omitted much, including how Poincaré duality is used in combination with turning the manifold upside down (replace Morse function f by $-f$), to convert handles of index i into handles of index $(n-i)$, so that one only ever needs to deal with handles of index $i \le (n+1)/2$. This is a crucial point in order to have enough extra dimensions when it comes to sliding handles around.