Topology around the Poincaré conjecture
The definition of the fundamental group involves the notion of a homotopy.

A homotopy $f_t: X \to Y$ between maps $f, g: X \to Y$ is a continuous function $f_t$ such that $f_0 = f$ and $f_1 = g$. The maps $f$ and $g$ are said to be homotopy equivalent.

Example: $X = [0,1]$, $Y = S^1$. The homotopy is given by the function $f_t(x) = \sin(2\pi t)x$.

The homotopy class of $f$ is denoted $[f]$. Two maps $f, g: X \to Y$ are homotopy equivalent if there is a homotopy between them.

The notion of a homotopy is fundamental in algebraic topology.
Theorem 3 (Differential-Topology Answer). Two manifolds are homotopy equivalent if and only if there is a continuous bijection between them, given a bijection (and a continuous inverse function) between them.

Corollary 2. A manifold is connected if and only if the group of its homotopy classes is trivial.

To prove that two manifolds are homotopy equivalent, one must construct a continuous bijection between them, and show that it is a homotopy equivalence. This can be done by exhibiting a homotopy equivalence between them, or by using other methods, such as the fundamental group or the homology groups.

For example, consider the real line and the circle. They are both one-dimensional manifolds, and there is a continuous bijection between them given by the function $f(x) = \tan(x/2)$, which is a bijection between the interval $(0, \pi/2)$ and the circle $S^1$. This function is a homotopy equivalence, and hence the real line and the circle are homotopy equivalent.

In particular, one can determine whether two manifolds are homotopy equivalent or not.

Theorem 3 (Differential-Topology Answer). Two manifolds $X$ and $Y$ are homotopy equivalent if and only if there is a continuous bijection between them, with a continuous inverse function.

Corollary 2. A manifold $X$ is connected if and only if the group of its homotopy classes $\pi_0(X)$ is trivial.

To prove that two manifolds $X$ and $Y$ are homotopy equivalent, one must construct a continuous bijection $f: X \to Y$ and show that it is a homotopy equivalence. This can be done by exhibiting a homotopy equivalence $g: Y \to X$ between them, or by using other methods, such as the fundamental group or the homology groups.

For example, consider the real line $\mathbb{R}$ and the circle $S^1$. They are both one-dimensional manifolds, and there is a continuous bijection $f: \mathbb{R} \to S^1$ given by $f(x) = \tan(x/2)$, which is a bijection between the interval $(0, \pi/2)$ and the circle $S^1$. This function $f$ is a homotopy equivalence, and hence $\mathbb{R}$ and $S^1$ are homotopy equivalent.

In particular, one can determine whether two manifolds are homotopy equivalent or not.

Theorem 3 (Differential-Topology Answer). Two manifolds $X$ and $Y$ are homotopy equivalent if and only if there is a continuous bijection between them, with a continuous inverse function.

Corollary 2. A manifold $X$ is connected if and only if the group of its homotopy classes $\pi_0(X)$ is trivial.
THE WHEELED MANifold. Construct the Wheeled Manifold.

**Figure 1. Constructing the Wheeled Manifold.**

\[
\begin{align*}
\mathbb{R}^n & \ni x \\
\mathbb{R}^m & \ni y
\end{align*}
\]

**Theorem 2. (Wheeled Manifold). There is a non-compact manifold.**

In dimension 2, two manifolds are homeomorphic if they are disjointly homotopy equivalent.

**Theorem 3. (Wheeled Manifold.)**

If two manifolds are homotopy equivalent, they are homeomorphic.

**Lemma 4.**

If two manifolds are homotopy equivalent, they are homeomorphic.

---

The wheeled manifold is constructed as an intersection union of sets that are homeomorphic.
Given \( W \subseteq \mathbb{R}^n \), show that for \( A \) and \( B \) as in the following.

\[
0 \neq x, \quad f(x) = \begin{cases} \|x\| / x \cdot \|x\| & \text{if } 0 < \|x\| < 1, \\ 0 & \text{if } \|x\| = 1 \end{cases}
\]

**Proof**

Theorem 6 (Alexander-Tychonoff). Every homotopy \( h : D \to \mathbb{R}^n \) extends to a homotopy \( H : D \to \mathbb{R}^n \) with \( H|_{\partial D} = h|_{\partial D} \).

This follows from the fact that \( h(\partial D) \) is homotopic to \( \partial D \), and the derivative of \( h \) is continuous in the following sense:

\[
\frac{d}{dt} h(t) = \begin{cases} \frac{d}{dt} h(t) & \text{if } t < 1, \\ 0 & \text{if } t = 1 \end{cases}
\]

If \( h \) is a continuous map from \( D \) to \( \mathbb{R}^n \), then \( h|_{\partial D} \) is homotopic to \( \partial D \).

**Observation.** \( \mathcal{S} \subseteq \mathbb{R}^n \) is the union of two manifolds with boundary.
homotopy. If $\Omega S^p$ is the space of based loops in $S^p$, then $\pi_1 \Omega S^p \cong \mathbb{Z}^p$. In particular, the homotopy groups of spheres are non-trivial, and one can use this fact to construct interesting homotopy classes of maps.

The key idea in understanding the homotopy of spheres is the concept of the suspension. For a space $X$, the suspension $\Sigma X$ is formed by taking the quotient of $X \times [0,1]$ under the relation $(x,t) \sim (x,1-t)$ when $t \neq 0$, and $(x,0) \sim (y,1)$. The suspension of a sphere $S^p$ is denoted $S^{p+1}$.

We now describe the process of attaching another handle to $S^n$. Consider the Mayer-Vietoris sequence for the pair $S^n \setminus S^m = E \times [0,1]$, where $E$ is a $k$-sphere. Then we have

$$\cdots \to H_{k+2}(S^n \setminus S^m) \to H_k(E) \to H_k(S^n \setminus S^m) \to H_{k+1}(S^n \setminus S^m) \to \cdots$$

The boundary map $H_k(E) \to H_{k+1}(S^n \setminus S^m)$ is non-trivial, and this allows us to attach a new handle to $S^n$.

**Figure 1:** A 2-handle that cancels a 1-handle.

**Figure 2:** A 2-handle decomposition of $S^3$.
The definitions and properties of the group $G$ are provided in the text. The group is generated by elements $g_i$ for $i = 1, 2, ..., n$. The group operation is defined as $g_i g_j = g_k$, where $k$ is the index of the element that is the result of the operation. The identity element is $g_0$. The inverse of an element $g_i$ is $g_{-i}$.

The action of the group $G$ on a space $X$ is defined by $g_i x = x'$. The action is transitive, meaning that for any two points $x, y$ in $X$, there exists an element $g_i$ such that $g_i x = y$.

The orbits of the action are the sets $\{g_i x : g_i \in G\}$, and the stabilizers are the sets $\{g_i \in G : g_i x = x\}$. The orbit-stabilizer theorem states that the number of elements in an orbit is equal to the index of the stabilizer in the group $G$.

The group $G$ is also a permutation group, meaning that it acts on a set of objects in a way that permutes the elements of the set. The group can be represented by a matrix $A = (a_{ij})$, where $a_{ij}$ is the number of times element $g_i$ is mapped by $g_j$.