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Automorphisms of Free Groups Have Finitely Generated Fixed Point Sets

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An automorphism of a finitely generated free group F extends to a homeomorphism of the end completion $\hat{F}$ of F. The set of fixed points of this homeomorphism is finitely generated in a certain sense. In particular this implies that the subgroup of elements fixed by the automorphism is finitely generated in the usual sense.

The emphasis on $\hat{F}$ instead of F in this paper is analogous to Thurston's study of surface groups where measured laminations are studied instead of simple closed curves. The techniques in this paper arose from a study of the behaviour of an automorphism under iteration, that is, the dynamics of the automorphism; see [1]. I thank Bill Thurston for explaining some of his ideas to me for analysing automorphisms of free groups. In particular the result on bounded cancellation below is due to Thurston and Matt Grayson. Peter Scott asked whether the fixed subgroup is always finitely generated in [3], and Gersten first answered this question in [2] using quite different methods.

Notation. F denotes a finitely generated free group and f an automorphism of F. We choose a basis Λ of F once and for all, identifying F with the set of finite reduced words in the elements of Λ and their inverses. $\hat{F}$ is the set of finite and infinite reduced words, i.e., sequences $\alpha_1\alpha_2\alpha_3 \cdots$, where $\alpha_i \in \Lambda$ or $\bar{\alpha}_i \in \Lambda$ and $\alpha_i \neq \bar{\alpha}_{i+1}$. $\hat{F}$ is the *end completion* of F. The *word length* of an element $\beta \in F$ with respect to Λ is written $|\beta|$, and the *word metric* on F is given by $d(\beta_1, \beta_2) = |\bar{\beta}_1\beta_2|$ for $\beta_1, \beta_2 \in F$. The *initial segment* of length $k \geqslant 0$ of $\beta \in \hat{F}$ is written $\operatorname{init}(\beta, k)$, and if $\beta \in F$ we require that $k \leqslant |\beta|$. If β_1 is an initial segment of β_2 we write $\beta_1 \leqslant \beta_2$. This puts a partial ordering on $\hat{F}$. The longest initial segment common to $\beta_1, \beta_2 \in \hat{F}$ is denoted $\beta_1 \wedge \beta_2$. The *initial segment metric* τ on $\hat{F}$ is defined by $r(\beta_1, \beta_2) = (1 + |\beta_1 \wedge \beta_2|)^{-1}$. With this metric $\hat{F}$ is compact and F is dense in $\hat{F}$.

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The *size* of an automorphism f is

$$S(f) = \max_{\alpha \in A} \{|f\alpha|, |f^{-1}\alpha|\}.$$

PROPOSITION. *An automorphism $f: F \to F$ extends uniquely to a homeomorphism $\hat{f}: \hat{F} \to \hat{F}$.*

Proof. Given $\alpha \in \hat{F}$ let $\alpha_p = \text{init}(\alpha, p)$ and $\beta_p = \bar{\alpha}_{p-1}\alpha_p$; thus $\alpha_p = \beta_1 \beta_2 \cdots \beta_p$ and $\beta_i \in A$ or $\bar{\beta}_i \in A$. Now $f(\alpha_{p+1}) = f(\alpha_p) f(\beta_{p+1})$ and $|f\alpha_p| \geqslant |\alpha_p|/S(f)$. Let m be the integer part of $|\alpha_p|/S(f)$; then since $|f\beta_{p+1}| \leqslant S(f)$, $\text{init}(f\alpha_{p+1}, m - S(f)) = \text{init}(f\alpha_p, m - S(f))$. It follows that $\text{init}(f\alpha_{p+t}, m - S(f))$ is independent of t for $t \geqslant 0$, and so f has a continuous extension given by $\hat{f}\alpha = \lim_{p \to \infty} f\alpha_p$. This extension is unique because F is dense in $\hat{F}$. Furthermore $\widehat{(f^{-1})} \circ \hat{f} = 1$, so $\hat{f}$ is a homeomorphism. ∎

BOUNDED CANCELLATION. Given an automorphism $f: F \to F$, there is a positive integer $C(f)$ such that if $\alpha_1, \alpha_2 \in F$ and $|\alpha_1 \alpha_2| = |\alpha_1| + |\alpha_2|$ then $|f(\alpha_1 \alpha_2)| \geqslant |f\alpha_1| + |f\alpha_2| - C(f)$.

Remark. $C(f)$ is called the *cancellation bound* for f and can be shown to be no more than $[S(f)]^2$.

Proof. $|f\alpha_1| + |f\alpha_2| - |f(\alpha_1\alpha_2)| = 2\,|f\bar{\alpha}_1 \wedge f\alpha_2|$ and f^{-1} is uniformly continuous on $\hat{F}$ so

$$\begin{aligned} \exists N > 0 \quad & |f\bar{\alpha}_1 \wedge f\alpha_2| > N \\ & \Rightarrow |\bar{\alpha}_1 \wedge \alpha_2| \geqslant 1 \\ & \Rightarrow |\alpha_1\alpha_2| < |\alpha_1| + |\alpha_2|. \quad ∎ \end{aligned}$$

COROLLARY. *Given $\alpha \in \hat{F}$ and $\alpha' \leqslant \alpha$ $\exists \beta \in F$ with $|\beta| \leqslant C(f)$ and $(f\alpha')\beta \leqslant f\alpha$.*

Proof. Write $\alpha = \alpha'\gamma$; then $(f\alpha')(f\gamma) = f\alpha$ so by bounded cancellation $|f\alpha \wedge f\alpha'| \geqslant |f\alpha'| - C(f)$. ∎

LEMMA. *Given $\alpha_1, \alpha_2 \in \hat{F}$ suppose $\alpha_1 \leqslant f\alpha_1$ and $\alpha_2 \leqslant f\alpha_2$. Let $\alpha = \alpha_1 \wedge \alpha_2$; then $f\alpha = \alpha\beta$ with $|\beta| \leqslant [2 + S(f)] \max\{C(f), C(f^{-1})\}$.*

Proof. Applying the corollary, for $i = 1, 2$ we have $\exists \gamma_i \in F$, $|\gamma_i| \leqslant C(f)$ and $(f\alpha)\gamma_i \leqslant f\alpha_i$. Let $k = |f\alpha| - |\alpha|$; then

$$|\alpha| \geqslant \min_i |(f\alpha) \wedge \alpha_i| \geqslant |f\alpha| - |\gamma_i| \geqslant k + |\alpha| - C(f),$$

hence $k \leqslant C(f)$. Now write $\alpha_i = \alpha\,\delta_i$ and observe that

$$\begin{aligned}|\alpha| &= |(f\alpha)(f\,\delta_1) \wedge (f\alpha)(f\,\delta_2)| \\ &\leqslant |f\alpha| + |(f\,\delta_1) \wedge (f\,\delta_2)|\end{aligned}$$

so $|(f\,\delta_1) \wedge (f\,\delta_2)| \geqslant -k$. Applying f^{-1} gives $|f^{-1}((f\,\delta_1) \wedge (f\,\delta_2))| > -k/S(f)$ and the corollary applied to f^{-1} with $(f\,\delta_1) \wedge (f\,\delta_2) \leqslant f\,\delta_i$ gives $|f^{-1}((f\,\delta_1) \wedge (f\,\delta_2)) \wedge \delta_i| \geqslant -k/S(f) - C(f^{-1})$, thus $|\delta_1 \wedge \delta_2| \geqslant -k/S(f) - C(f^{-1})$. But $|\delta_1 \wedge \delta_2| = 0$ because $\alpha = \alpha_1 \wedge \alpha_2$, thus $k \geqslant -S(f)\,C(f^{-1})$. Hence $|k| \leqslant S(f)M$, where $M = \max\{C(f), C(f^{-1})\}$.

Thus $|\,|(f\alpha)\gamma_i| - |\alpha|\,| \leqslant [S(f)+1]M$ but $(f\alpha)\gamma_1$ and α are both initial segments of $f\alpha_1$ and so $\exists \beta' \in F$ with $|\beta'| \leqslant [S(f)+1]M$ and $(f\alpha)\gamma_1\beta' = \alpha$, thus $|\beta| = |\gamma_1\beta'| \leqslant [S(f)+2]M$. ∎

Let A be a subset of $\hat{F}$, the set *generated* by A is defined to be the smallest closed set C containing A which is also closed under multiplication and taking inverses whenever this makes sense, i.e.,

$$\alpha, \beta \in C \qquad \text{and} \qquad |\alpha| < \infty \;\Rightarrow\; \bar{\alpha}, \qquad \alpha\beta \in C.$$

A subset of $\hat{F}$ is *finitely generated* if it is generated by a finite subset of $\hat{F}$. Given an automorphism f of F, the *fixed point set* is

$$\mathrm{fp}(f) = \{\alpha \in \hat{F} \colon \hat{f}(\alpha) = \alpha\}$$

and the *fixed subgroup* is

$$\mathrm{fgp}(f) = \{\alpha \in F \colon f(\alpha) = \alpha\}.$$

THEOREM. *The fixed point set of an automorphism of a finitely generated free group is finitely generated.*

COROLLARY 1. *The fixed subgroup of such an automorphism is finitely generated.*

COROLLARY 2. *If the fixed subgroup of f contains only the identity, then the fixed point set is finite.*

Conjecture. Is there an upper bound on the minimum number of elements of $\hat{F}$ needed to generate $\mathrm{fp}(f)$, depending only on $\mathrm{rank}(F)$?

Proof of Theorem. First we prove that $\mathrm{fgp}(f)$ is finitely generated (as a group). Suppose not; then choose any basis of $\mathrm{fgp}(f)$

$$\alpha_1, \alpha_2, \alpha_3, \ldots$$

and define a new basis of fgp(f) by

$$\beta_1 = \alpha_1$$

$$\beta_i = \text{an element of minimum word length in } \langle a_1, ..., \alpha_{n-1} \rangle \alpha_n,$$

where word length is measured (as always) relative to Λ. Since $\hat{F}$ is compact, there is a convergent subsequence $\beta_{n_i} \to \gamma \in \hat{F}$. We may suppose that $r(\beta_{n_j}, \gamma) < r(\beta_{n_i}, \gamma)$ whenever $i < j$. Continuity of $\hat{f}$ implies that $\hat{f}\gamma = \gamma$. Define $\delta_i = \beta_{n_i} \wedge \gamma$ and note that $i \leqslant j$ implies $\delta_i \leqslant \delta_j$. Now $\beta_{n_i} \leqslant f(\beta_{n_i})$ and $\gamma \leqslant f(\gamma)$ so by the lemma, $f(\delta_i) = \delta_i \varepsilon_i$ with $|\varepsilon_i| \leqslant M$ for some constant M. There are only finitely many elements of F with length less than or equal to M, so for some $j > i$, $\varepsilon_j = \varepsilon_i$. It follows that $\eta = \delta_i \bar{\delta}_j$ is fixed by f. Express η as a word in the α_i as $\eta = \alpha_{p_1}^{\pm 1} \alpha_{p_2}^{\pm 1} \cdots \alpha_{p_m}^{\pm 1}$. Let $p = \max_k p_k$ and choose $n_k > \max(p, n_j)$. Then certainly $\eta\beta_{n_k}$ is fixed by f, and $\eta \in \langle \alpha_1, \alpha_2, ..., \alpha_{n_k - 1} \rangle \alpha_{n_k}$ so by definition of β_{n_k} we must have that $|\eta\beta_{n_k}| \geqslant |\beta_{n_k}|$. However, $\delta_j \leqslant \delta_k$ so that $|\eta\beta_{n_k}| \leqslant |\beta_{n_k}| - |\delta_j| + |\delta_i|$, but $|\delta_j| > |\delta_i|$, giving a contradiction.

To complete the proof, choose a minimal set of elements of $\hat{F}$ which together with fgp(f) generate fp(f). Call these elements $\alpha_1, \alpha_2, \alpha_3, ...$ and define a new set of words by

$$\beta_i = \text{an element in } \langle \text{fgp}(f) \rangle\, \alpha_i \text{ maximising } r(\beta_i, \text{fgp}(f)).$$

When $|\alpha_i| < \infty$ this is equivalent to minimising word length; however, in this situation $|\alpha_i| = \infty$. Apply the preceding argument to obtain η and β_{n_k}. Choose an integer m large enough that if $\gamma = \text{init}(\beta_{n_k}, m)$ then $|\gamma \wedge \text{fgp}(f)| = |\beta_{n_k} \wedge \text{fgp}(f)|$. [The notation $|\alpha \wedge \text{fgp}(f)|$ for $\alpha \in F$ means $\max\{|\alpha \wedge \beta| : \beta \in \text{fgp}(f)\}$.] Then

$$|(\eta\gamma) \wedge \text{fgp}(f)| = |(\eta\beta_{n_k}) \wedge \text{fgp}(f)|$$

because $\eta \in \text{fgp}(f)$. As before, $|\eta\gamma| < |\gamma|$, and $|\gamma \wedge \text{fgp}(f)| = |(\eta\gamma) \wedge \text{fgp}(f)| + |\gamma| - |\eta\gamma|$; thus

$$r(\beta_{n_k}, \text{fgp}(f)) = r(\gamma, \text{fgp}(f)) < r(\eta\gamma, \text{fgp}(f)) = r(\eta\beta_{n_k}, \text{fgp}(f)).$$

This contradicts the choice of β_{n_k} as before. ∎

References

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