

COMPLEXITY FUNCTIONS ON 1-DIMENSIONAL COHOMOLOGY

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For a smooth, closed n -manifold M , we define an upper semi-continuous integer-valued complexity function on $H^1(M; \mathbb{R})$ using Morse theory. This measures how far an integral class is from being a fiber of a fibration. The fact complexity minimisers are open generalises Tischler's result on the openness of classes dual to fibrations. We then use this to define a complexity function on 1-dimensional cohomology of a finitely presented group, which is constant on open rays from the origin and vanishes precisely on the geometric invariant due to Bieri, Neumann and Strebel.

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1. INTRODUCTION

This work has its origin in a desire to exhibit extra structure in the non-fibred faces of the unit norm ball of the Thurston norm on first homology of a 3-manifold (see [14]). However, our constructions are more general. To keep the discussion brief, we restrict ourselves to closed manifolds and finitely presented groups in this note. Throughout this paper we regard 0 as a natural number in \mathbb{N} and manifolds are smooth.

Suppose M is a connected oriented closed n -manifold. The *Morse complexity function*

$$\tilde{m} : H^1(M; \mathbb{R}) \longrightarrow \mathbb{N}^{n+1}$$

is a vector, where the k^{th} component counts the number of critical points of Morse index k of a minimal closed Morse 1-form representing the cohomology class, with the minimum taken over all representatives using lexicographic order. Since \tilde{m} is constant on rays, it gives a function on the sphere

$$m : S(M) = (H^1(M; \mathbb{R}) \setminus \{0\}) / \mathbb{R}_+ \longrightarrow \mathbb{N}^{n+1}.$$

Then $m(\psi) = \vec{0}$ if and only if ψ is represented by a non-singular closed 1-form. It follows from work of Tischler [15] (see Fried [5, Theorem 1]) that

the set of cohomology classes of non-singular closed 1-forms is an open cone in $H^1(M; \mathbb{R}) \setminus \{0\}$, which is non-empty if and only if there is an integral class represented by a codimension-1 manifold that is a fiber of a fibration of M over S^1 .

THEOREM 1. *Suppose M is a closed orientable n -manifold. For each $\vec{m} \in \mathbb{N}^{n+1}$ there is an open subset $U \subset S(M)$ such that $m(\psi) \leq \vec{m}$ if and only if $\psi \in U$.*

COROLLARY 2. *Suppose M is a closed orientable n -manifold, and $S \subset M$ is a 2-sided connected non-separating codimension-1 submanifold, and X is the compact n -manifold obtained by deleting an open tubular neighbourhood of S . Then $\partial X = S_- \sqcup S_+$ and X is the union of $S_- \times [0, 1]$ and some handles. Given a primitive class $\phi \in H^1(M; \mathbb{Z})$, let $h(\phi)$ be the minimum number of handles for such S dual to ϕ . For each $k \geq 0$ there is an open subset $U \subset S(M)$, such that $h(\phi) \leq k$ if and only if $[\phi] \in U$.*

In Section 3, we define an analogous complexity function, called the *tropical rank*, for a finitely presented group G by considering all manifolds with fundamental group isomorphic to G . We then re-interpret this function using HNN decompositions of G and relate it to the geometric invariant $\Sigma(G)$ due to Bieri, Neumann and Strebel. To state the properties of this function, let $S(G)$ be the set of equivalence classes of non-trivial homomorphisms $\varphi: G \rightarrow \mathbb{R}$, where two such homomorphisms are equivalent if they are positive scalar multiples of each other. If $\varphi(G)$ is discrete, then $\text{rank}[\varphi]$ is *roughly* the minimum number of generators that must be added to the amalgamating subgroup A to obtain B in an HNN extension $G \cong B*_A$ corresponding to φ .

THEOREM 3. *Let G be a finitely presented group. The tropical rank*

$$\text{rank} : S(G) \rightarrow \mathbb{N}$$

is upper semi-continuous and has image bounded above by the minimal number n of generators of G . Moreover, for rational classes $[\varphi] \in S\mathbb{Q}(G)$ we have

- (1) $\text{rank}[\varphi] \leq n - 1$,
- (2) $\text{rank}[\varphi] = 0$ if and only if $[\varphi] \in \Sigma(G)$, and
- (3) $\text{rank}[\varphi] = 0$ and $\text{rank}[-\varphi] = 0$ if and only if $\ker \varphi$ is finitely generated.

Most of the results in this paper are known. However we have collected those results that seem most relevant to low dimensional topology and geometric group theory with a view to providing a concise source. Our focus is on *handles* in manifolds and *generators* for groups rather than Novikov homology, and is more in the spirit of the Thurston norm and BNS invariant. We end this introduction by discussing open questions about the special case of 3-manifolds.

Suppose M is a closed, orientable 3-manifold. There is a natural identification $S(M) = S(\pi_1(M))$. The first Morse rank on $S(M)$ gives an upper bound on the tropical rank on $S(\pi_1(M))$, and we do not know if this is always an equality. Thurston [14] defined a semi-norm on $H^1(M; \mathbb{R})$. Each primitive class $c \in H^1(M; \mathbb{Z})$ can be represented by an embedded, orientable surface $F \subset M$. If F is connected, define $\chi_-(F) = \max\{0, -\chi(F)\}$, and otherwise $\chi_-(F) = \sum \chi_-(F_i)$, where the sum ranges over its connected components. Then the semi-norm of c is the infimum of $\chi_-(F)$ taken over all surfaces F representing c . Thurston showed that the unit ball of this semi-norm is a finite sided polytope, and that the complexity zero classes correspond to the interiors of top-dimensional faces of this polytope, called the *fibred faces*. In particular, the set of cohomology classes of non-singular integral classes has image that is dense in an open *polyhedral* subset of $S(M)$. Apart from this, we do not know how the first Morse rank and the tropical rank are related to the polyhedral structure of the Thurston norm ball.

2. MANIFOLDS

Let M be a closed n -manifold. We show that elements of $H^1(M; \mathbb{R})$ are represented by Morse 1-forms. This is used to define a complexity taking values in \mathbb{N}^{n+1} (using lexicographic order) for a first cohomology class. Nearby cohomology classes are represented by nearby Morse 1-forms with the *same* singularities. This implies the set of classes with complexity less than some value is open. We then relate this complexity to handle decompositions of M and the Novikov numbers of M . The notation and terminology in this section follows Farber [4], and most of the results are surely known to the experts.

2.1. The Morse complexity

Let M be a closed, n -manifold. A smooth function $f : M \rightarrow \mathbb{R}$ is *Morse* if at every critical point of f the Hessian $d^2 f$ is non-singular. This Hessian is a quadratic form

$$-(x_1^2 + \cdots + x_i^2) + (x_{i+1}^2 + \cdots + x_n^2)$$

of index i . A Morse function gives a handle decomposition of M with i -handles corresponding to critical points of index i .

Let ω be a closed 1-form on M . For every open ball $U \subset M$, there is a smooth function $f_U : U \rightarrow \mathbb{R}$ satisfying $\omega|_U = df_U$ and f_U is determined uniquely by ω up to a constant. In local coordinates, we have:

$$\omega_x = \sum a_i(x) dx_i$$

and hence $a_i(x) = \frac{\partial f}{\partial x_i}(x)$. It follows that the zeros of ω in U are precisely the critical points of f_U . A zero of ω is called *non-degenerate* if the critical point of the corresponding function is a Morse singularity, and the index of the singularity is termed the *Morse index* of the zero of ω . The form ω is termed *Morse* if all of its zeros are non-degenerate.

For a Morse 1-form ω denote by $m_i = m_i(\omega)$ the number of zeros of Morse index i . This is finite since M is compact. This gives a *complexity vector* $\tilde{m}(\omega) = (m_0, \dots, m_n)$. We use lexicographic ordering on these vectors so $c < d$ if there is $i \geq 0$ with $c_i < d_i$ and $c_j = d_j$ for all $j < i$.

By Theorem 10 every element $\xi \in H^1(M; \mathbb{R})$ is represented by a Morse 1-form ω and we define $\tilde{m}(\xi)$ to be the minimum of $\tilde{m}(\omega)$ over such ω . Note that $\tilde{m}(\xi) = \tilde{m}(r\xi)$ for $r > 0$, however in general $\tilde{m}(\xi) \neq \tilde{m}(-\xi)$, so \tilde{m} descends to a well-defined function on the sphere,

$$m : S(M) = (H^1(M; \mathbb{R}) \setminus \{0\})/\mathbb{R}_+ \longrightarrow \mathbb{N}^{n+1},$$

called the *Morse complexity function*. Note that $S(M) \cong S^{k-1}$, where $k = \text{rank } H^1(M; \mathbb{R})$.

In the case $\xi = 0$ a minimising Morse 1-form is $\omega = df$ for some Morse function f on M . Then $\tilde{m}(0)$ gives the least number of handles in a decomposition of M using lexicographic ordering. In particular $\tilde{m}(0) = (1, 0, \dots, 0, 1)$ if and only if M is *diffeomorphic* to a sphere. The existence of exotic 7-spheres implies that $\tilde{m}(0)$ is not an invariant of homotopy type.

Instead of minimising the whole complexity vector, one may also be interested in minimising any one of its component functions. For $\xi \in H^1(M; \mathbb{R})$, the *i -th Morse rank* is

$$\text{m-rank}_i(\xi) = \min_{[\omega]=\xi} m_i(\omega).$$

We again have $\text{m-rank}_i(\xi) = \text{m-rank}_i(r\xi)$ for $r > 0$, however in general $\text{m-rank}_i(\xi) \neq \text{m-rank}_i(-\xi)$. We therefore view m-rank_i as an integer-valued function on the sphere $S(M)$, and note that in general, the i -th Morse rank is not the i^{th} component of the Morse complexity function.

PROPOSITION 4. *The i -th Morse rank*

$$\text{m-rank}_i : S(M) \rightarrow \mathbb{N}$$

is upper semi-continuous for each $i \in \{0, 1, \dots, n\}$. That is, for each $x \in S(M)$, there is an open neighbourhood $N(x)$ with respect to the usual topology of S^{k-1} such that $\text{m-rank}_i(y) \leq \text{m-rank}_i(x)$ for all $y \in N(x)$.

Since $S(M)$ is compact, it follows that the range of $\text{m-rank}_i : S(M) \rightarrow \mathbb{N}$ is finite. Since the function is upper semi-continuous, we have the following immediate consequence:

COROLLARY 5. *The set Σ_i of all points of minimal i -th Morse rank is open in $S(M)$. Moreover, rational points are dense in Σ_i .*

Remark 6. For all rational points, we have $\text{m-rank}_0[\xi] = 0 = \text{m-rank}_n[\xi]$ since they can be represented by circle valued Morse functions. However, Corollary 5 does not imply that

$$\text{m-rank}_0 \equiv 0 \equiv \text{m-rank}_n .$$

Are there examples, where the functions m-rank_0 and m-rank_n are not trivial?

Example 7 (The torus). Consider the torus $T^k = V/\mathbb{Z}^k$, where $V = \mathbb{R}^k$. A linear map $L \in V^*$ gives an exact 1-form dL on \mathbb{R}^k , which is preserved by translations, and therefore is the pullback of a closed 1-form ω_L on T^k . Since there are no singularities, m-rank_1 vanishes identically.

Example 8. There are manifolds M with the property that $\text{m-rank}_1 : S(M) \rightarrow \mathbb{N}$ does not vanish identically but is zero on a dense set; for instance 3-manifolds with each face of the Thurston norm ball a fibred face, such as (zero-framed surgery on) the complements of the Whitehead link or the Borromean rings (see [14]).

Example 9 (Surfaces). Let F be a surface of genus $g \geq 1$, and $\xi \in H^1(F, \mathbb{Z})$ be a non-trivial primitive element. Then ξ is dual to a circle valued Morse function $F \rightarrow S^1$ with no critical points of indices 0 or 2. Since critical points of index 1 can only be cancelled against those of index 0 or 2, the Euler characteristic of F implies $m(\xi) = (0, 2g - 2, 0)$ and $\text{m-rank}_1(\xi) = 2g - 2$.

2.2. Proofs of Theorem 1 and Proposition 4

Endow M with a Riemannian metric. We let

$$B(x; r) = \{y \in M \mid d(x, y) < r\}.$$

If $B(x; r)$ is called an *open ball*, then it is understood that r is chosen such that $B(x; r)$ is diffeomorphic to $\text{int } B^n \subset \mathbb{R}^n$. This condition holds for r sufficiently small. At each $x \in M$, we have a well-defined real number,

$$\|\omega\|_x = \sqrt{\langle \omega_x, \omega_x \rangle_x},$$

arising from the inner product at x . We have $\|\omega\|_x = 0$ if and only if x is a zero of ω . The *norm* of ω is $\|\omega\| = \sup \|\omega\|_x$. A closed 1-form ω defines a de-Rham cohomology class $[\omega] \in H^1(M; \mathbb{R})$.

It is well known that the set of Morse functions is dense in the set of smooth functions. Lalonde, McDuff and Polterovich [8, Lemma 5.1] show that

the harmonic representative of a non-trivial class in $H^1(M, \mathbb{R})$ is a Morse 1-form for a generic Riemannian metric. We give an elementary proof of the density of Morse forms in every cohomology class. Another proof is in [12, Ch.2 Theorem 1.25].

THEOREM 10. *If M is a closed Riemannian manifold, then the set of Morse 1-forms in each cohomology class is dense.*

Proof. We must show that given $\epsilon > 0$ and a closed 1-form ω , there is a Morse 1-form, η , such that $[\eta] = [\omega] \in H^1(M; \mathbb{R})$ and $\|\omega - \eta\| < \epsilon$.

Suppose ψ is a closed 1-form on M and $U \subset M$ is an open set such that $\psi|_{\bar{U}}$ is Morse, and $\psi|_{\bar{U}}$ has finitely many zeros which are all in U . Suppose $B := B(x; r) \subset M$ is an open ball contained in larger open balls $2B := B(x; 2r)$ and $3B = B(x; 3r)$. Given $\delta > 0$ we show below that there is a 1-form ψ_1 on M such that $\psi = \psi_1$ outside $3B$ and $\|\psi_1 - \psi\| < \delta$ and $\psi_i|_{\text{cl}(U \cup B)}$ is Morse with finitely many singularities, which are all in $U \cup B$. Since ψ_1 and ψ are equal outside the contractible set $3B$ it follows they represent the same cohomology class.

Assuming this, there is a finite cover of M by balls $B_i := B(x_i; r) \subset M$ with $1 \leq i \leq m$, each contained in balls $B(x_i; 2r) \subset B(x_i; 3r)$. We start with $\psi_0 = \omega$ and $U_0 = \emptyset$ and $\delta = \epsilon/m$ and inductively apply the above to produce a 1-form ψ_k which is Morse on $U_k = \cup_{i=1}^k B_i$. Then $\eta = \psi_m$ is as required.

It remains to prove the claim. Since ψ has finitely many zeros in U , there is an open set V with $\bar{V} \subset U$ and V contains all the zeros of $\psi|_{\bar{U}}$. Set $\mu = \inf \|\psi\|_x$ where the infimum is over x in the compact set $\bar{U} \setminus V$. Then $\mu > 0$.

The closed sets $C = \text{cl}(B \setminus U)$ and $D = \text{cl}(M \setminus 2B) \cup \text{cl}(V)$ are disjoint. Thus, there is a smooth function $\lambda : M \rightarrow [0, 1]$ with $\lambda(C) = 1$ and $\lambda(D) = 0$. Let $K = 1 + \|\text{d}\lambda\|$.

Since ψ is closed and $3B$ is a ball, integrating ψ along paths starting at the base point gives a smooth function $g : 3B \rightarrow \mathbb{R}$ with $\text{d}g = \psi$ on $3B$. Since Morse functions are dense in $C^\infty(\text{int}(B^n))$, there is a Morse function $h : 3B \rightarrow \mathbb{R}$ with $\sup |g - h| < \mu/3K$ and $\|\text{d}(g - h)\| < \mu/3$.

The function $f : 3B \rightarrow \mathbb{R}$ given by $f = g + \lambda \cdot (h - g)$ equals h on C and g on $3B \cap (V \cup (3B \setminus 2B)) \subset D$. Thus, f is Morse on $3B \cap (C \cup V)$. We have

$$\text{d}f = \text{d}g + (h - g) \text{d}\lambda + \lambda \text{d}(h - g)$$

There are no critical points of f in $3B \cap (\bar{U} \setminus V)$ since on this set $\|\text{d}f\| > \mu/3$ because $\|\text{d}g\| \geq \mu$ and $\|\text{d}\lambda\| \leq K$ so $\|(h - g) \text{d}\lambda\| < \mu/3$ and $\|\lambda \text{d}(h - g)\| \leq \|\text{d}(h - g)\| < \mu/3$.

Putting this together, f is Morse on the subset of $3B$ contained in $C \cup V \cup (\bar{U} \setminus V) = C \cup \bar{U} = B \cup \bar{U}$.

Observe that ψ and df are equal on $3B \setminus 2B$. Thus, we may define ψ_1 to be ψ on $M \setminus 2B$ and df on $3B$. Then ψ_1 is Morse on $3B \cap (B \cup \bar{U})$ because $\psi_1 = df$ there. It is Morse on $\bar{U} \setminus 2B$ because it is ψ there. Hence, ψ_1 is Morse on $\bar{U} \cup B$ as required. \square

The next result implies that nearby homology classes are represented by nearby 1-forms.

LEMMA 11. *If M is a closed Riemannian manifold and $\epsilon > 0$, there is a neighbourhood $U \subset H^1(M; \mathbb{R})$ of 0 with the property that for every $\phi \in U$, there is a closed 1-form η with $[\eta] = \phi$ and $\|\eta\| < \epsilon$.*

Proof. We first prove the result for the k -torus $T^k = V/\mathbb{Z}^k$, where $V = \mathbb{R}^k$. As noted in Example 7, a linear map $L \in V^*$ gives an exact 1-form dL on \mathbb{R}^k , which is preserved by translations, and therefore is the pullback of a closed 1-form ω_L on T^k . The map $V^* \rightarrow \Omega^1(T^k)$ given by $L \mapsto \omega_L$ is continuous and $V^* \rightarrow H^1(T^k; \mathbb{R})$ given by $L \mapsto [\omega_L]$ is an isomorphism.

For the general case let $k = \beta_1(M)$. There is a map $f : M \rightarrow T^k$, which induces an isomorphism $f^* : H^1(T^k; \mathbb{R}) \rightarrow H^1(M; \mathbb{R})$. Then $f^*(\omega_L)$ is a closed 1-form on M , which varies continuously with L and this gives an isomorphism $V^* \rightarrow H^1(M; \mathbb{R})$. \square

Next is a local stability result: nearby cohomology classes are represented by Morse 1-forms with the *same* zeros.

LEMMA 12. *Given a closed Morse 1-form ω and $\delta > 0$, there is a neighbourhood $U \subset H^1(M; \mathbb{R})$ of 0 with the property that if $\alpha \in U$, then there exists a Morse 1-form η with $[\eta] = \alpha + [\omega]$ and $\|\omega - \eta\| < \delta$ and there is an open set $A \subset M$, which contains all the zeros of η and of ω and $\omega|_A = \eta|_A$. In particular $\tilde{m}(\eta) = \tilde{m}(\omega)$.*

Proof. Let ω be a Morse 1-form. Then ω has a finite number of zeros, say z_1, \dots, z_m . For each z_i choose a radius $r_i > 0$ such that $B_i = B(z_i; r_i)$, $2B_i = B(z_i; 2r_i)$ and $3B_i = B(z_i; 3r_i)$ are open balls. We may do this so the balls $3B_i$ are pairwise disjoint. Define $A = \bigcup B_i$ and $3A = \bigcup 3B_i$ and set

$$\delta = \inf\{\|\omega\|_x : x \in M \setminus A\}.$$

Then $\delta > 0$ because $M \setminus A$ is compact. Choose a smooth function $\lambda_i : M \rightarrow [0, 1]$ with $\lambda_i(B_i) = 1$ and $\lambda_i(M \setminus 2B_i) = 0$. Choose $\epsilon > 0$ smaller than $\delta/3$ and $(1/3)(1 + \|d\lambda_i\|)^{-1}$ for every i .

Let $U \subset H^1(M; \mathbb{R})$ be the neighbourhood of 0 given by Lemma 11. Given $\alpha \in U$ there is a closed 1-form ϕ representing α with $\|\phi\| < \epsilon$. By Theorem 10 we may assume ϕ is Morse.

Since $\|\phi\| < \delta$ all the zeros of $\eta = \omega + \phi$ are contained in A . Below we construct a Morse 1-form η' which equals η in $M \setminus 3A$ so $[\eta'] = [\eta] = \alpha + [\omega]$. Moreover $\eta' = \omega$ on A completing the proof.

To construct η' we modify η in $B = B_i \subset 3B_i = 3B$ as follows. There are smooth functions $f, g : 3B \rightarrow \mathbb{R}$ with $\omega = df$ and $\phi = dg$ on $3B$. Set

$$h = f + \lambda_i(g - f)$$

Then $h = g$ on B and $h = f$ on $3B \setminus 2B$. A calculation as in the proof of Theorem 10 shows h is Morse and with one singular point at z_i . Define η' to be η in $M \setminus 2B$ and dh in $3B$. On the overlap $3B \setminus 2B$ these are both η , and $\eta = \omega$ on B . Since the balls $3B_i$ are pairwise disjoint these modifications can be done independently. \square

Proofs of Theorem 1 and Proposition 4. The statements follow directly from the local stability result (Lemma 12) together with the following two facts. First, the topology on $S(M)$ induced by the norm agrees with the usual topology. Second, for each class $\xi \in H^1(M; \mathbb{R})$, one may choose a representative ω_1 with the property that $\tilde{m}(\omega_1) = m(\xi)$, and a representative ω_2 with the property that $\text{m-rank}_i(\omega_2) = \text{m-rank}_i(\xi)$. \square

2.3. The i -th handle rank of a rational class

There is a topological description of the Morse complexity and the Morse ranks for rational classes. Suppose $(N, \partial_-N, \partial_+N)$ is a *cobordism*. Thus, N is a compact connected n -dimensional manifold with $\partial N = \partial_-N \sqcup \partial_+N$. Suppose \mathcal{H} is a handle decomposition of N given by attaching a finite number of handles to a collar $\partial N_- \times [0, 1]$ of ∂_-N . Such a handle decomposition determines a vector

$$\hat{h}(N, \partial_-N, \mathcal{H}) = (h_0, h_1, \dots, h_n) \in \mathbb{N}^{n+1},$$

where h_i is the number of handles of index i . The *handle complexity* of N rel ∂_-N is the vector

$$\tilde{h}(N, \partial_-N) = \min_{\mathcal{H}} \hat{h}(N, \partial_-N, \mathcal{H}),$$

where the minimum is taken with respect to lexicographical ordering over all such handle decompositions. Thus, $\tilde{h}(N, \partial_-N) = 0$ if and only if N is diffeomorphic to $\partial N_- \times [0, 1]$.

Similarly, the i^{th} *handle rank* of N rel ∂_-N is

$$(2.1) \quad \text{h-rank}_i(N, \partial_-N) = \min_{\mathcal{H}} h_i.$$

A primitive element $\beta \in H_{n-1}(M; \mathbb{Z})$ is represented by a compact, connected, 2-sided submanifold $V \subset M$ with tubular neighbourhood $\nu(V)$. An

orientation of M determines a *transverse orientation* to V and hence a cobordism $(N = M \setminus \nu(V), V_-, V_+)$. The minimum of $\check{h}(N, V_-)$ over such V is the *handle complexity* $h(\beta)$ of β , and the minimum of $\text{h-rank}_i(N, V_-)$ is the i^{th} *handle rank* $\text{h-rank}_i(\beta)$ of β . In general $h(-\beta) \neq h(\beta)$ so the choice of orientation on M is necessary.

Finally, if $\xi \in H^1(M; \mathbb{Z})$ is primitive, denote the Poincaré dual to ξ by $D_M(\xi) \in H_{n-1}(M; \mathbb{Z})$. Then define

$$h(\xi) = h(D_M(\xi)) \quad \text{and} \quad \text{h-rank}_i(\xi) = \text{h-rank}_i(D_M(\xi))$$

to be the handle complexity and i^{th} handle rank of ξ respectively.

LEMMA 13. *Suppose M is a smooth closed oriented manifold. For each primitive class $\xi \in H^1(M; \mathbb{Z})$, we have $h(\xi) = m(\xi)$ and $\text{h-rank}_i(\xi) = \text{m-rank}_i(\xi)$.*

Proof. This follows from the well-known correspondence between Morse functions on M and cobordisms associated to regular level sets (see [10, 11]). \square

Remark 14. The group rank, which is defined below, gives a lower bound on the first handle rank, and hence on the first Morse rank (see Remark 20).

2.4. Lower bounds from Novikov numbers

For every first cohomology class $\xi \in H^1(M; \mathbb{R})$ and every $k \in \mathbb{N}$, there is a *Novikov Betti number* $b_k(\xi)$ and a *Novikov torsion number* $q_k(\xi)$ (see [4], §1.5 for three equivalent definitions). For $\xi = 0$, $b_k(\xi)$ is the usual k^{th} Betti number of M and $q_k(\xi)$ is the minimal number of generators of the torsion subgroup of $H_k(M; \mathbb{Z})$. For general ξ , the ring \mathbb{Z} is replaced with a local system of modules over a fancier ring. The Novikov inequalities (see [4], §2.3) state that

$$(2.2) \quad m_k(\omega) \geq b_k(\xi) + q_k(\xi) + q_{k-1}(\xi),$$

if $[\omega] = \xi$. Moreover, $b_k(\xi) = b_k(r\xi)$ for every $0 \neq r \in \mathbb{R}$ and $q_k(\xi) = q_k(r\xi)$ for every $r > 0$. It follows that for $[\xi] \in S(M)$, we have

$$(2.3) \quad \text{m-rank}_k[\xi] \geq b_k(\xi) + q_k(\xi) + q_{k-1}(\xi).$$

Moreover, $b_0(\xi) = 0 = q_0(\xi)$, so the relationship is particularly nice when $k = 1$:

$$(2.4) \quad \text{m-rank}_1[\xi] \geq b_1(\xi) + q_1(\xi).$$

We call a function $f: S(M) \rightarrow \mathbb{N}$ *polyhedral* if each superlevel set $f^{-1}[k, \infty)$ is a spherical polytope. Farber ([4], §1.6) gives a nice description of the geometry of the Novikov inequalities and, in particular, a polyhedrality result. Since the Morse rank has this polyhedral lower bound, this triggers the following question:

Question 15. Is the Morse rank a polyhedral function?

Example 16 (3-manifolds with arbitrarily large Morse rank). Farber ([4], §3.4.2) gives a family of closed 3-manifolds X_n , which are obtained as the connected sum of $S^2 \times S^1$ and the 0-surgery on the connected sum of n trefoil knots, and satisfy $H^1(X_n; \mathbb{Z}) \cong \mathbb{Z}^2$. Moreover, the generator ξ supported by the $S^2 \times S^1$ summand has the property that every Morse closed 1-form representing it has at least n zeros of index 1. This is established using inequalities arising from a ring, which is more elaborate than the Novikov ring (see [4], Theorem 3.6).

3. GROUPS

Let G be a finitely presented group, and $S(G)$ be the set of equivalence classes of non-trivial homomorphisms $\varphi: G \rightarrow \mathbb{R}$, where two such homomorphisms are equivalent if they are positive scalar multiples of each other. Then $S(G) \cong S^{n-1}$, where $G/G' \cong \mathbb{Z}^n \oplus \text{torsion}$. Define the *tropical rank* of the class $[\varphi] \in S(G)$ by

$$\text{rank}[\varphi] = \min_M \{\text{m-rank}_1[\xi]\},$$

where the minimum is taken over all smooth closed manifolds M with $\pi_1(M) = G$, and where ξ is the image of φ under the canonical identification

$$\text{Hom}(\pi_1(M), \mathbb{R}) \cong H^1(M; \mathbb{R}).$$

The tropical rank is a well-defined function $S(G) \rightarrow \mathbb{N}$, since $r\xi$ is the image of $r\varphi$ for $r > 0$ and since m-rank_1 is integer valued. The definition involves taking the minimum, and so Proposition 4 directly implies:

PROPOSITION 17. *The tropical rank is upper semi-continuous and has bounded image.*

The word *tropical* alludes to the fact that this rank gives information on the complement of the Bieri-Neumann-Strebel invariant [1]—this complement was characterised by Brown [3] using group theoretic valuations, and hence can be viewed as a tropical set.

The above proposition gives the first part of Theorem 3. The remaining parts concern rational classes and follow from the equality of Morse rank and handle rank (Lemma 13) together with the equality of handle rank and a *group rank* defined using HNN extensions (Lemma 19), fundamental facts from [1] (summarised in Lemma 18) and an application of Magnus rewriting (Lemma 22).

3.1. The Bieri-Neumann-Strebel invariant and HNN extensions

We now define the Bieri-Neumann-Strebel invariant $\Sigma(G)$. Choose a finite generating set for G and denote \mathcal{G} the corresponding Cayley graph, with the convention that G acts from the left on this graph. Given the non-trivial homomorphism $\varphi: G \rightarrow \mathbb{R}$, define a G -equivariant map $\tilde{\varphi}: \mathcal{G} \rightarrow \mathbb{R}$ by sending vertices to their images under φ and extending linearly over edges. Denote \mathcal{G}_φ the maximal subgraph of \mathcal{G} contained in $\tilde{\varphi}^{-1}(-\infty, 0]$. It turns out that the connectedness of \mathcal{G}_φ is independent of the representative of $[\varphi]$ and the generating set for G . Define $\Sigma(G) \subseteq S(G)$ to be the set of precisely those $[\varphi]$ for which \mathcal{G}_φ is connected.¹ It is shown in [1] that $\Sigma(G)$ is an open subset of $S(G)$.

If the non-trivial homomorphism $\varphi: G \rightarrow \mathbb{R}$ has discrete (and hence cyclic) image, then it is a positive scalar multiple of a unique epimorphism $G \rightarrow \mathbb{Z}$, and it is termed a *discrete* homomorphism. The point $[\varphi] \in S(G)$ is called *rational*. The set

$$SQ(G) = \{[\varphi] \in S(G) \mid \varphi \text{ is discrete}\}$$

of rational points is dense in $S(G)$.

Suppose G is a finitely presented group with epimorphism $\varphi: G \rightarrow \mathbb{Z}$. An *associated HNN-extension* of (G, φ) is (B, A, t, α) , where:

$$(3.1) \quad G = \langle t, B \mid A = t^{-1}\alpha(A)t \rangle,$$

and $A \subseteq B \subseteq \ker \varphi$, both A and B are finitely generated, $\alpha: A \rightarrow B$ is a monomorphism, and $t \in G$ with $\varphi(t) = 1$. (A proof of this fact can be found in [2]; we give an independent proof in §3.3.) In this case, (G, φ) is said to *split over* A . If $A = B$, then the HNN-extension is called *ascending* in [1] (see the discussion in §4.1 of [6]). The following two facts are established in [1]:

- (1) $[\varphi] \in \Sigma(G)$ if and only if φ corresponds to an ascending HNN extension;
- (2) $\ker \varphi$ is finitely generated if and only if $[\varphi] \in \Sigma(G)$ and $[-\varphi] \in \Sigma(G)$.

In [6], the complexity of $[\varphi]$ is defined as the minimal rank of the group A , where the minimum is taken over all HNN-extensions of G of the form (3.1), and this gives a measure analogous to the Thurston norm [14]. We will show that the tropical rank defines a complementary complexity, namely the *minimal difference* between A and B . This will be established through two alternative viewpoints—using HNN extensions and handle decompositions.

¹The definition of the BNS invariant involves various conventions and choices of sign, and one often encounters a definition using the preimage of $[0, \infty)$ instead of $(-\infty, 0]$. The above definition matches [1, 6].

3.2. The group-rank

Suppose G is a finitely presented group with epimorphism $\varphi: G \rightarrow \mathbb{Z}$. For an associated HNN extension (3.1), define $\text{g-rank}(B, A)$ to be the minimal number of elements one needs to add to A in order to generate B , i.e.

$$\text{g-rank}(B, A) = \min_{B=\langle b_1, \dots, b_n, A \rangle} n.$$

Then define

$$(3.2) \quad \text{g-rank}(\varphi) = \min(\text{g-rank}(B, A)),$$

where the minimum is taken over all HNN-extensions of G of the form (3.1).

The above stated facts from [1] concerning the Bieri-Neumann-Strebel invariant give the following characterisation of classes with trivial group rank:

LEMMA 18. *We have:*

- (1) $\text{g-rank}(\varphi) = 0$ if and only if $[\varphi] \in \Sigma(G)$, and
- (2) $\text{g-rank}(\varphi) = 0$ and $\text{g-rank}(-\varphi) = 0$ if and only if $\ker \varphi$ is finitely generated.

3.3. The handle-rank

Define

$$(3.3) \quad \text{h-rank}(\varphi) = \min_M \{\text{h-rank}_1(\xi)\},$$

where the minimum is taken over all closed n -manifolds M with $\pi_1(M) = G$, and where ξ is the image of φ under the canonical identification $\text{Hom}(\pi_1(M), \mathbb{Z}) \cong H^1(M; \mathbb{Z})$.

We wish to compare the handle rank with the group rank. Note that the groups A and B in presentation (3.1) are not necessarily finitely presentable (according to the Higman embedding theorem they are recursively presentable). We work around this issue using “fake HNN-extensions” as follows.

Suppose G is a finitely presented group and $\varphi: G \rightarrow \mathbb{Z}$. There is a finite presentation of G of the form

$$(3.4) \quad \langle s, d_i, c_j \mid r_k(d_1, \dots, c_1, \dots) = 1, s c_j s^{-1} = w_j(d_1, \dots, c_1, \dots) \rangle$$

with $\varphi(s) = 1$ and $\varphi(d_i) = \varphi(c_j) = 0$. The existence of this presentation can be seen either algebraically or topologically.

Algebraically, one can apply Magnus re-writing to any presentation of G to convert it to the desired form (see [9, §IV.5] and also the proof of Lemma 22).

The topological argument starts by choosing a closed, connected manifold M with $\pi_1(M) = G$. Then $\varphi: G \rightarrow \mathbb{Z}$ determines a map $f: M \rightarrow S^1$. Let

$p \in S^1$ be a regular value with connected level set $H = f^{-1}(p)$, and $N = M \setminus \nu(H)$ be the complement of an open regular neighbourhood of H in M . Then $\partial N = H_+ \cup H_-$ is the disjoint union of two copies of H and there are two (not necessarily injective) inclusion homomorphisms $\alpha_{\pm}: \pi_1(H) \rightarrow \pi_1(N)$. Applying the Generalised Van Kampen Theorem [7, Theorem 6.2.11] to $M \setminus H$ and $\nu(H)$ gives a presentation of $\pi_1(M) = G$ of the form

$$\langle s, \pi_1(N) \mid s\alpha_+(h)s^{-1} = \alpha_-(h) \quad \forall h \in \pi_1(H) \rangle,$$

where $\pi_1(N)$ and $\pi_1(H)$ are finitely presented groups since both N and H are compact. Since $\pi_1(H)$ is finitely generated we only need the relations $s\alpha_+(h)s^{-1} = \alpha_-(h)$ for a generating set. This gives a presentation in the form (3.4) as follows. Denote the generators and relators of $\pi_1(N)$ by d_i and r_k respectively and for each relation of the form $s\alpha_+(h)s^{-1} = \alpha_-(h)$ add one generator c_j and the relations $c_j = \alpha_+(h)$ and $sc_j s^{-1} = \alpha_-(h)$ with $\alpha_{\pm}(h)$ expressed as words in the generators of $\pi_1(N)$. This completes the topological argument that a presentation of the form (3.4) exists.

From (3.4), one obtains an HNN extension as follows. Let $A = \langle c_j \rangle \leq G$ and $B = \langle d_i, c_j \rangle \leq G$. Then A and B are finitely generated (but possibly not finitely presented) subgroups of G with $A \leq B$. Moreover, $sAs^{-1} \leq B$ by construction, so the conjugation map $\alpha: a \mapsto sas^{-1}$ takes A to an isomorphic subgroup of B . Whence $G = \langle s, B \mid A = s^{-1}\alpha(A)s \rangle$ and we also have

$$\text{g-rank}(B, A) \leq | \{d_i\} |.$$

LEMMA 19. $\text{h-rank}(\varphi) = \text{g-rank}(\varphi)$

Proof. Each relative handle decomposition gives rise to a fake HNN-decomposition of the form (3.4). For any φ , there is a manifold M with relative handle decomposition (N, H_-, \mathcal{H}) realising the minimum in (3.3). Using the above notation with $B = \text{im}(\pi_1(N) \rightarrow \pi_1(M))$ and $A = \text{im}(\pi_1(H_-) \rightarrow \pi_1(M))$ gives:

$$\text{g-rank}(\varphi) \leq \text{g-rank}(B, A) \leq | \{d_i\} | = \text{h-rank}_1(N, H_-) = \text{h-rank}(\varphi).$$

To prove $\text{h-rank}(\varphi) \leq \text{g-rank}(\varphi)$, build a 2-complex with fundamental group G as follows. Let $G = \langle t, B \mid tAt^{-1} = \alpha(A) \rangle$ be an HNN-extension realising minimal group rank for φ . Choose finite generating sets $\{a_i\}$ for A and $\{a_i, b_j\}$ for B such that $| \{b_j\} | = \text{g-rank}(\varphi)$. Since G is a finitely presentable group and generated by $\{t, a_i, b_j\}$, there is a finite presentation of G with these generators and of the form

$$(3.5) \quad G = \langle t, a_i, b_j \mid r_k(a_1, \dots, b_1, \dots), ta_it^{-1} = w_i(a_1, \dots, b_1, \dots) \rangle.$$

Given the finite presentation, we let C be the free group in $\{a_i\}$ and D be the group

$$\langle a_i, b_j \mid r_k(a_1, \dots, b_1, \dots) \rangle.$$

Build a 2-complex, Y_0 , with one 0-cell and 1-cells for the *elements* b_i, a_i and $\alpha(a_i)$, and 2-cells for the relators in the presentation of D and the extra relations $\alpha(a_i)w_i^{-1}$, where w_i is a word in the a_i and b_i equal to the element $\alpha(a_i)$. This 2-complex has fundamental group D .

Then take the sub-complex $X \subset Y_0$ formed by the cells for C and cross it with an interval, giving a 2-complex, $X \times I$. Let $Y_1 = Y_0 \cup_{X \times \{0\}} (X \times I)$. Then Y_1 still has fundamental group D .

Let Y_2 be the 3-complex obtained from Y_1 as follows. For each a_i , identify the 1-cell representing a_i in $X \times \{1\} \subset Y_1$ with the 1-cell representing $\alpha(a_i)$. It follows that Y_2 has fundamental group G .

Embed Y_2 in \mathbb{R}^5 and embed \mathbb{R}^5 in \mathbb{R}^6 in the standard way, taking $\mathbb{R}^5 \ni x \mapsto (x, 0) \in \mathbb{R}^6$. Let Y_3 be the boundary of a regular neighbourhood of the embedding. This is a 5-manifold and it has a product region, identified with $W \times (0, 1)$ for suitable W , corresponding to the subcomplex $X \times (0, 1)$. Identify $W = W \times \{\frac{1}{2}\} \subset Y_3$.

We claim that there is a handle decomposition of $\overline{Y_3 \setminus W}$ relative to $W_- \subset \partial(\overline{Y_3 \setminus W})$, $W_- \cong W$, with exactly one 1-handle for each b_i . To show this, remove the interior of each 1-handle corresponding to each b_i , and denote the result Z . Then

$$\pi_1(W_-) \rightarrow \pi_1(Z) \rightarrow \pi_1(Z, W_-) \rightarrow \pi_0 = 0,$$

whence $\pi_1(W_-) \cong \pi_1(Z)$ and it follows from [13, Lemma 6.15] (and the remark after that lemma), that no additional 1-handles are needed. Moreover, W_- is connected as a consequence of the chosen embedding of Y_2 into \mathbb{R}^6 , and it is non-separating due to the presentation of the fundamental group. Hence, we have shown that $\text{h-rank}(\varphi) \leq \text{g-rank}(\varphi)$. \square

Remark 20. If $\Gamma = \pi_1(M)$ and φ corresponds to ξ , then $\text{g-rank}(\varphi) \leq \text{h-rank}_1(\xi)$, giving another lower bound for Morse rank.

Example 21. Example 7 implies that the tropical rank for the group $\mathbb{Z}^2 = \pi_1(T)$ vanishes identically. For rational classes, this is easy to verify using HNN extensions, since any epimorphism $\mathbb{Z}^2 \rightarrow \mathbb{Z}$ is equivalent to the epimorphism $\varphi: \langle a, b \mid a^{-1}ba = b \rangle \rightarrow \mathbb{Z}$ with $\varphi(a) = 1$ and $\varphi(b) = 0$, whence $\text{rank}[\varphi] = 0$.

3.4. Consequences for the tropical rank

Recall that if $\varphi: G \rightarrow \mathbb{R}$ is discrete, then $\varphi(G)$ is infinite cyclic, and hence there is a unique $r \in \mathbb{R}_+$ such that $r\varphi: G \rightarrow \mathbb{Z}$. It follows from Lemmata 13 and 19 that:

$$\text{rank}[\varphi] = \text{h-rank}(r\varphi) = \text{g-rank}(r\varphi).$$

LEMMA 22. *If G has a presentation with n generators, then $\text{rank}[\varphi] \leq n$ for each $[\varphi] \in S(G)$, and $\text{rank}[\varphi] \leq n - 1$ for each rational class $[\varphi] \in \text{SQ}(G)$.*

Proof. We first address the general case. As in the proof of Lemma 19, we can construct a closed manifold M with a handle decomposition having exactly n 1-handles in its handle decomposition. We can define a Morse 1-form on M dual to this handle decomposition (see [10, 11]). This has exactly n critical points of index 1. The local stability result Lemma 12 implies that this is a universal bound on the number of critical points of index 1 of every Morse 1-form on M .

Next suppose that $[\varphi] \in \text{SQ}(G)$. Given the presentation $G = \langle g_i \mid r_j \rangle$ with n generators we apply Magnus rewriting as follows. If $\varphi(g_{i_0}) = \pm 1$ for the generator g_{i_0} of G , then choose $s = g_{i_0}^{\pm 1}$ and $d_i = g_i s^{\mp \varphi(g_i)}$ for all $i \neq i_0$. We have $\varphi(s) = 1$ and $\varphi(d_i) = 0$ for each d_i . Now rewrite the relators using this new generating set $\{s, d_i\}$. If s appears in the relator r_k , then (up to cyclic permutation) there is a substring in r_k of the form $s^{-1}ws$, where w is a word in the d_i . We introduce a new generator c_j , the relation $sc_j s^{-1} = w$ and replace the substring $s^{-1}ws$ by c_j in r_k . Note that $\varphi(c_j) = 0$. Iterating this procedure results in the desired presentation, and the number of generators d_i is bounded above by $n - 1$.

Hence, assume $\varphi(g_i) \neq \pm 1$ for all generators of G . Since φ is an epimorphism, there are two generators g_{i_0} and g_{i_1} such that their respective images are coprime. So there are $p, q \in \mathbb{Z}$ such that φ maps $s = g_{i_0}^p g_{i_1}^q$ to one. Introduce this new generator, and replace each g_i by $b_i = g_i s^{-\varphi(g_i)}$. This gives a new generating set $\{s, b_i\}$ with $n + 1$ elements. Rewrite all relators in these generators and add the relator $s^{-1}(b_{i_0} s^{\varphi(g_{i_0})})^p (b_{i_1} s^{\varphi(g_{i_1})})^q$. The latter lets us choose either $c_{j_0} = b_{i_0}$ or $c_{j_1} = b_{i_1}$, and we put $d_i = b_i$ for all remaining indices. With this we proceed as above, again obtaining a generating set $\{s, c_j, d_i\}$ with at most $n - 1$ generators d_i . \square

Question 23. Is the tropical rank a polyhedral function?

Remark 24. The definition of tropical rank took into account only the 1-handles (*generators*) since in general, the groups arising in an HNN extension are finitely generated but not necessarily finitely presented. For groups with the property that every finitely generated subgroup is finitely presented, one can also take into account the 2-handles (*relators*), giving a finer \mathbb{N}^2 -valued complexity, for which one can define analogous \mathbb{N}^2 -valued group and handle ranks, which agree with it on rational classes.

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