Contents

An Undetected Slope in a Knot Manifold

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0. Introduction

The work of Cũller and Shalen **[CS]** using representation varieties to construct incompressible surfaces via degeneration has been seminal; it is still the only general technique available. However, it raises a natural question: Which surfaces are so obtained?

In order to formalize this question, we proceed as follows. Suppose that M is a knot complement, that is to say, a compact 3-manifold with a single torus boundary component. Choose coordinates α, β for $\pi_1(\partial M)$; an embedded incompressible, ∂-incompressible surface $(F, \partial F)$ gives rise to a collection of parallel simple closed loops in ∂M. Choose one such and write the homology class it represents as $\alpha^a\beta^b$; then to $(F, \partial F)$ we associate the rational number a/b (possibly $\infty = 1/0$) which we call the ***boundary slope*** of $(F, \partial F)$. A theorem of Hatcher **[H]** implies that M has only a finite number of boundary slopes.

There are several natural notions in this context, which we now define. We will say that a surface $(F, \partial F)$ is ***detected*** if there is a sequence $\{\rho_n\}$ of representations of the fundamental group of M into $\mathrm{SL}(2, \mathbb{C})$, which lies on a curve inside the character variety of $\pi_1(M)$ and converges to an ideal point, so that if one looks at the action of $\pi_1(M)$ on the tree coming from this sequence the group $\pi_1(F)$ is conjugate into an edge stabiliser.

A boundary slope a/b is ***detected*** if there is a sequence of representations $\{\rho_n\}$ as above with:

(a) The traces $\mathrm{tr}(\rho_n(\alpha^a\beta^b))$ remain bounded as $n \to \infty$.
(b) There is some homology class γ in $\pi_1(\partial M)$ for which $\mathrm{tr}(\rho_n(\gamma))$ does not remain bounded as $n \to \infty$.

The results of **[CS]** imply that many slopes are detected. However, as an example, a fibre of a fibration over S^1 cannot be detected by irreducible representations, but the slope of a fibre may be detected because of the presence of another, non-fibre surface which also has this slope. This happens with the knot 8_{20}. A fibre can be detected if we allow abelian representations.

In this note we give an example which shows that for knots in rational homology spheres it is possible for rather complicated surfaces to be undetected:

Theorem 1.5. *There is a knot complement in a rational homology sphere containing a boundary slope of* $1/6$ *which is not detected.*

Our example is also surprising in the sense that although this slope is not detected by representations into $\mathrm{SL}(2, \mathbb{C})$, it is detected by representations into $\mathrm{PSL}(2, \mathbb{C})$, if one uses some appropriate notions of degeneration; for example that of **[B]**.

Before describing the example, we make some remarks of a general nature. Suppose that a 3-manifold M has two boundary components each of which is a torus. Then there are maps between character varieties

$$i_j : X(M) \longrightarrow X(T_j), \quad j = 1,2$$

induced by restriction to the boundary. There are now two phenomena we wish to understand. Firstly, the image of one or both of the i_j 's may not have complex dimension 2. It seems to be a hard and interesting problem to say when this does or does not happen, but we shall not explore this question here. The second problem is that even if the image is two dimensional, there may be curves which are missed out of the image. Accordingly, if $\dim(i_j(X(M))) = \dim(X(T_j)) = 2$, any component of $i_j(X(M)) \setminus X(T_j)$ of complex dimension 1 we call a ***forbidden curve***.

We observe for example that for a Brunnian link it is always the case that there are two forbidden curves given by $\mathrm{tr}(\mu) = \pm 2$. The special property of the Whitehead link which we exploit is that it has a third forbidden curve coming from the presence of an immersed punctured Klein bottle.

Here is an informal description of the example. Let W be the complement in S^3 of the Whitehead link. By consideration of the representations of the fundamental group $\pi_1(W)$ into $\mathrm{SL}(2,\mathbb{C})$ we find that the simple closed curve C of slope 2 on one of the boundary tori has the property that there is a curve of representations of $\pi_1(W)$ which send C to $+\mathrm{Id}$ but only finitely many which send C to $-\mathrm{Id}$; that is to say, $\mathrm{tr}(C) = -2$ gives a forbidden curve.

The loop D of slope 6 on the trefoil has the property that every irreducible representation of its fundamental group must send D to $-\mathrm{Id}$; so that if we glue these manifolds together in a way which sends C to D, then the possibilities for representations of the resulting manifold are severely limited.

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1. The example

In this section we outline the construction. Of necessity, this involves some calculations; some of these are relegated to an appendix.

Our example is built from the complement of the Whitehead link in S^3; we denote this manifold by W. It has two torus boundary components which can be exchanged by a symmetry. Fix some labelling once and for all of these tori as T_0 (i.e. "outer") and T_i ("inner"). The (meridian, longitude) pairs we denote by (μ_0, λ_0) and (μ_i, λ_i) respectively.

Following **[CCGLS]**, one can compute the $\mathrm{SL}(2,\mathbb{C})$ "eigenvalue variety" of $\pi_1(W)$ (see the appendix), we denote this by $E(\pi_1(W))$. This is defined quite generally (see the appendix), but one can give an especially clear heuristic description for the Whitehead link, since it is a two bridge link. For one sees easily that $\pi_1(W)$ is generated by μ_0 and μ_i. Then given an irreducible representation ρ, we may conjugate so that the generators map

$$\mu_i \mapsto \begin{pmatrix} m_i & 1 \\ 0 & 1/m_i \end{pmatrix} \quad \text{and} \quad \mu_0 \mapsto \begin{pmatrix} m_0 & 0 \\ t & 1/m_0 \end{pmatrix}.$$

The group relation now imposes on m_0, m_i and t a single polynomial constraint $F(m_i, m_0, t) = 0$ which describes an affine algebraic set. (Of course this description is a mild simplification since there is more than one way of doing the preliminary conjugation.)

There are two projection mappings $i_0 \colon E(\pi_1(W)) \to E(\pi_1(T_0))$ and $i_i \colon E(\pi_1(W)) \to E(\pi_1(T_i))$ which come from restricting a representation to the boundary. The image of these maps describes which eigenvalue pairs are possible on the boundary tori. We use (m_0, ℓ_0) etc. as notation for the eigenvalue pairs.

Note that the image of either mapping cannot possibly fill up all of $E(\pi_1(T_0)) = \mathbb{C}^2$, since the Whitehead link is a pure link so that restricting $m_0 = \pm 1$ forces $\ell_0 = \pm 1$. Thus the image $i_0(E(\pi_1(W)))$ misses out most of the curves $m_0 = \pm 1$. Since we are dealing with eigenvalues, the image must also avoid the curves $m_0 = 0$ and $\ell_0 = 0$. We shall show explicitly in this example that there is another curve which is (largely) missed out; calculation of this curve is our next task.

In order to find what eigenvalues are possible on the boundary tori, we use elimination theory **[M]**; details are deferred to the appendix. The result is that we form polynomials in the eigenvalue triples (m_i, m_0, ℓ_i) and (m_i, m_0, ℓ_0). We call these the outer and inner polynomials denoting them $O(m_i, m_0, \ell_i)$ and $I(m_i, m_0, \ell_i)$ respectively.

As explained in the appendix, the crucial property of these polynomials is that given any triple of nonzero complex numbers (m_i, m_0, ℓ_0) with $O(m_i, m_0, \ell_0) = 0$ (or a triple (m_i, m_0, ℓ_i) with $I(m_i, m_0, \ell_i) = 0$), there is an irreducible representation of $\pi_1(W)$ whose restriction to the inner boundary torus has eigenvalue m_i for the meridian, and on the outer boundary torus has eigenvalues (m_0, ℓ_0) for the meridian-longitude pair.

Lemma 1.1. *If a representation of the Whitehead link complement has $m_0^2.\ell_0 = -1$, then m_0 is ± 1 or $\pm i$.*

Proof. The loop $C = \mu_0^2 \lambda_0$ can be expressed in terms of the generators μ_i, μ_0 of the fundamental group of the Whitehead link as a conjugate of

$$\mu_0(\mu_i \mu_0^{-1} \mu_i)\mu_0(\mu_i \mu_0^{-1} \mu_i)^{-1}.$$

Putting r $= (\mu_i \mu_0^{-1} \mu_i)$, we recognise this as the relator in the Klein bottle group. (It follows that there is an immersion of a punctured Klein bottle into the Whitehead link with boundary $\mu_0^2 \lambda_0$.)

Now suppose that we have a representation ρ of the Whitehead link for which $\mathrm{tr}(\rho(\mu_0^2 \lambda_0)) = -2$. If $\rho(\mu_0^2 \lambda_0)$ is parabolic, then it follows that ρ is parabolic on all of T_0. Alternatively if $\mu_0^2 \lambda_0 = -I$, then $\rho(r \mu_0 r^{-1}) = -\rho(\mu_0^{-1})$, and it follows that $\mathrm{tr}(\rho(\mu_0)) = 0$. We have shown that if $\mathrm{tr}(\rho(\mu_0^2 \lambda_0)) = -2$ then $\mathrm{tr}(\rho(\mu_0)) = 0, \pm 2$, but $\mu_0, \mu_0^2 \lambda_0$ is a basis of $\pi_1(T_0)$, from which it follows that there are only finitely many possibilities for the character of ρ restricted to T_0. □

Remark. An alternative proof comes from the fact that if one puts $\ell_0 = -1/m_0^2$ in $O(m_i, m_0, \ell_0)$ then the resulting polynomial has only the given roots together with the root $m_i = 0$, which is forbidden for a representation.

This is the special feature of the Whitehead link complement which we are about to exploit; a representation of $\pi_1(W)$ having $m_0^2\ell_0 = -1$ has only finitely many possibilities for m_0. (Remarkably, this fails to be true for the curve $m_0^2\ell_0 = +1$.) It would be interesting to have other hyperbolic examples.

We now return to the matter in hand:

Lemma 1.2. *Let F denote an incompressible, ∂-incompressible surface in the Whitehead manifold which is detected in $\mathrm{SL}(2,\mathbb{C})$. If the slope of F on the inner torus is $1/6$, the only slopes possible on the outer torus are 2, -24, 26. All these slopes occur.*

Proof. We recall from **[CCGLS]** that information concerning boundary slopes may be obtained from the Newton polygon. We briefly sketch the salient details. Given a polynomial f(x,y) we define its ***Newton polygon*** to be the convex hull of the set in the plane defined by: $\{(r,s)|x^r y^s$ has nonzero coefficient in $f(x,y)\}$.

Roughly, the edges of the Newton polygon describe ways of going to infinity in the affine variety f(x,y)=0 which keep some ratio $x^\alpha y^\beta$ bounded. Let us briefly recall how this works. For notational simplicity, suppose that the Newton polygon has a slope of zero and that all coefficients lie below this slope. Then we may write $f(x,y) = y^\beta P_\beta(x) + \sum_{r=o}^{\beta-1} y^r P_r(x)$ where the $P_j(x)$ are polynomials in x. Factoring out y^β we see that for very large y the term which dominates the behaviour of the equation $f(x,y) = 0$ is $P_\beta(x)$ and it follows that we may obtain points on the curve $f(x,y) = 0$ provided we choose x to be very close to a root of $P_\beta(x)$. In other words, there is a way of going to infinity along the curve defined by $f(x,y)$ which keeps x bounded, i.e., a slope of zero.

The precise result is the following:

Theorem [CCGLS]. *Suppose that a knot has eigenvalue variety defined by the polynomial $f(\lambda,\mu)$. Form a convex body in the $\lambda\mu$-plane by taking the convex hull of the points $\{(\lambda,\mu)|\lambda^r\mu^s$ has non zero coefficient in $f(\lambda,\mu)\}$. Then the slopes of faces of this convex body are boundary slopes of the knot.*

The above theorem has a natural interpretation for any curve of representations, even in the context of links; namely it describes what happens when one restricts a curve of representations to a boundary component. The reason for this is the following: If one projects such a curve into the $\mathbb{C}^2$ of possible eigenvalues associated with a boundary component, the complex dimension of the image can be at most one. If this image is a point, then the sequence is constant on the boundary component. If not, then (after possibly taking Zariski closure) we see a complex curve in $\mathbb{C}^2$. It is a general fact **[Ha**, Proposition 1.13] that a curve in $\mathbb{C}^2$ is the zero set of a single polynomial in two variables and therefore has associated to it a Newton polygon. Ideal points on this curve give rise to splittings of the group of the link and the boundary slopes on the given component give slopes of surfaces which can arise.

To prove 1.2, we seek to understand degenerations with the property that the class $\mu_i \cdot \lambda_i^6$ has bounded trace. If we have any curve of degenerations then the behaviour for the eigenvalues of the inner boundary component is coded by the polyonomial $I(m_i, m_0, \ell_i)$. The possibilities for the eigenvalues can be understood by replacing m_i by C/ℓ_i^6, where C is any function bounded away from 0 and ∞. Making this substitution in $I(m_i, m_0, \ell_i)$; this yields a polynomial which has Newton polygon given in Figure 1a.

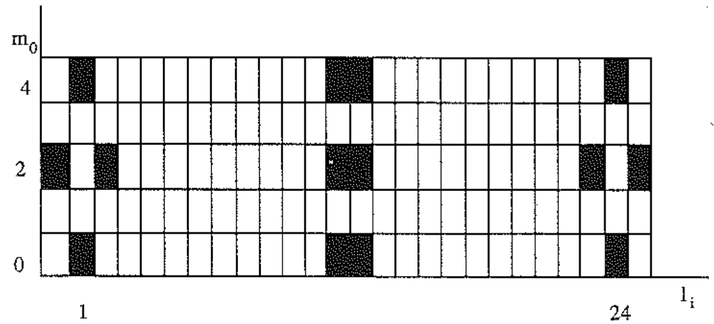

Figure 1a

A nonzero term $m_0^r\ell_i^s$ in the polynomial is indicated by a black rectangle in the s,r entry. Notice that coefficients of the extremal vertices are as indicated in Figure 1b.

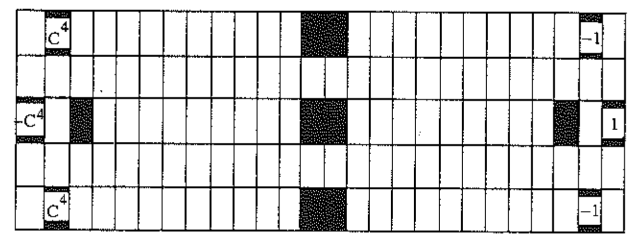

Figure 1b

The slopes in this picture are ± 2 and 0. The 0 slope corresponds to the case when ℓ_i is bounded away from ∞, that is to say, the slope on the inner torus is 0 which we have excluded in our hypothesis.

Analysis of the slopes of ± 2 now follows along exactly the lines of the first paragraph of this proof. Consider the slope of 2 coming from the top left of the Newton polygon. Since C is bounded away from ∞, two terms which dominate are $-C^4 m_0^2(1 - m_0^2\ell_i)$ so that we may go to infinity on the curve in a way which has $m_0^2\ell_i$ converging to 1. Similarly for the other slopes and we see that in all cases we have m_0^2 converges to $\ell_i^{\pm 1}$.

There are two cases, and making the substitutions $m_i = -1/\ell_i^6$ and $\ell_i = m_0^{\pm 2}$ in $O(m_i, m_0, \ell_0)$ we see that in either case, we obtain a polynomial which has Newton polygon indicated in Figure 2.

It follows we obtain the candidate slopes of 2, 26 and -24. Further, from the remarks made above all these boundary slopes do arise in W, completing the proof. □

Lemma 1.3. *Suppose that $(F_j, \partial F_j)$ is an incompressible, ∂-incompressible surface in a manifold $(M_j, \partial M_j)$ for $j = 1, 2$. Then gluing a component of ∂M_1 to a component of ∂M_2 so that ∂F_1 is glued to ∂F_2 yields an incompressible, ∂-incompressible surface.*

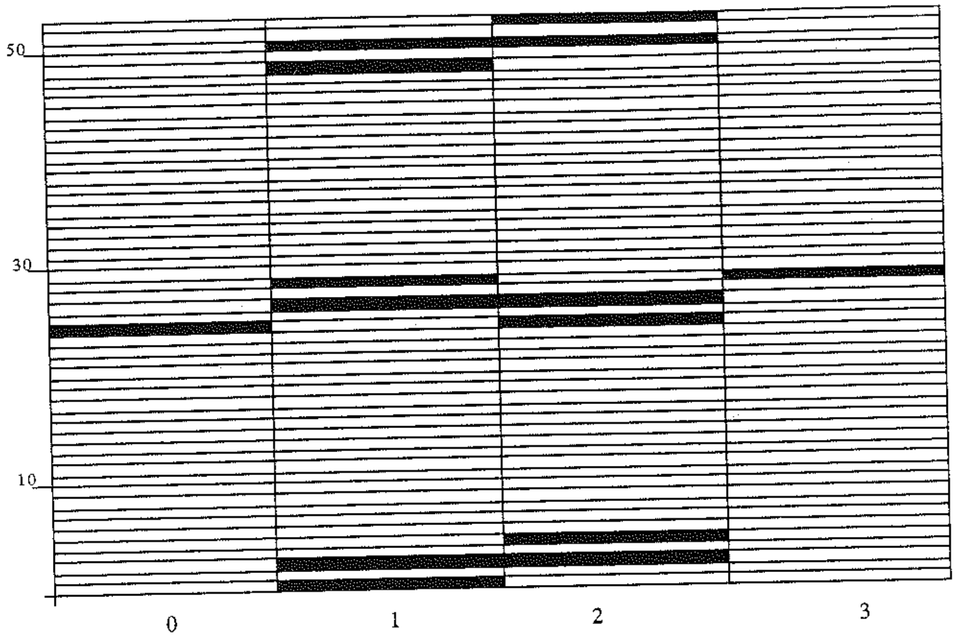

Figure 2

Proof. This follows from standard 3-manifold techniques. □

To fix our ideas, suppose that we have glued the outer torus of the Whitehead manifold to the exterior of the trefoil so that μ_0 is glued to the meridian of the trefoil μ_T and λ_0 to $\mu_T^4 \lambda_T$. The resulting manifold X then has $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}_4$. We continue to refer to the boundary torus of X as the inner torus, etc.

In this gluing we see that the loop of slope 2 on the outer torus is glued to the loop of slope 6 in the trefoil. The incompressible annulus of the trefoil has slope 6, and it follows that:

Corollary 1.4. *The manifold X contains an incompressible, ∂-incompressible surface U, of slope $1/6$.* □

Our main claim is:

Theorem 1.5. *The slope $1/6$ is not detected by degeneration of representations into* $\mathrm{SL}(2, \mathbb{C})$.

Proof. The crucial subclaim here is:

Subclaim. *The only degenerations of the Whitehead link group which are bounded on the outer torus produce slopes 0 or 4 on the inner torus.*

Proof of subclaim. Since the representation is bounded on the outer torus, m_0 is bounded away from 0 and ∞. We can therefore plot a Newton polygon for $I(m_i, m_0, \ell_i)$; see Figure 3.

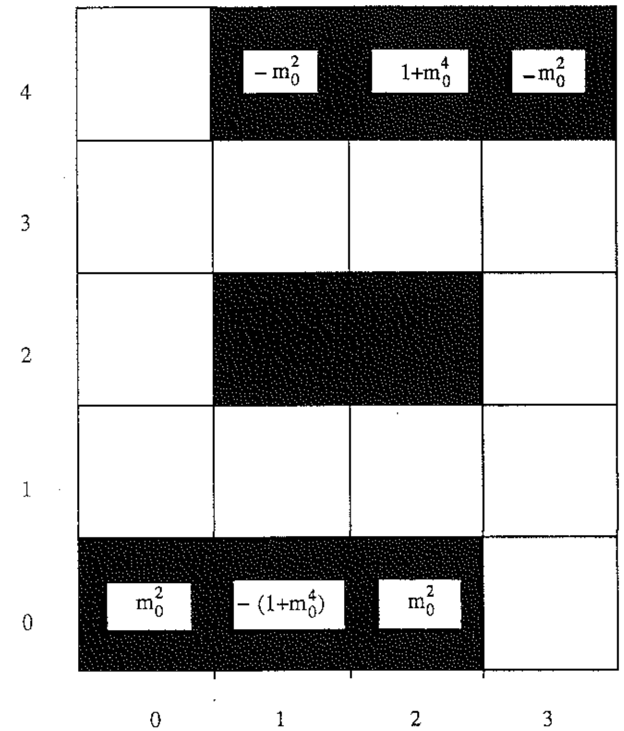

Figure 3

As above, since m_0 is bounded away from 0 and ∞ the only slopes possible on the inner torus are 0 and 4. This proves the subclaim. □

Suppose that this slope is detected, that is to say, there is a degeneration ρ_n of $\pi_1(X)$, so that the representations are blowing up on the boundary torus of X. There are two possibilities:

Case A. The representations ρ_n restrict to abelian representations of the trefoil complement. For such representations, the longitude of the trefoil has eigenvalue 1. We claim that this cannot be the only loop on the splitting torus with bounded trace, since the longitude of the trefoil is glued to the loop on the outer torus $\mu_0^{-4}\lambda_0$, and -4 was not in the list of possibilities of Lemma 1.2. Therefore the degeneration is bounded on the outer torus. But the subclaim implies that this produces slopes of 0 or 4 on the inner torus, a contradiction.

Case B. The representations are nonabelian on the trefoil. Then a calculation shows that the curve of slope 6 on the trefoil is mapped to Id by every such representation. Since this curve is glued to the curve of slope 2 on the outer torus, we see via Lemma 1.1 that only finitely many representations are possible on the splitting torus; whence the ρ_n are bounded and we again obtain a slope of 0 or 4 on the outer torus.

In either case, we obtain a contradication, completing the proof. □

Remark. Using the techniques of **[CCGLS]**, one can compute the A-polynomial for this knot; its Newton polygon has slopes 0, 4 and ± 1; these latter arising from representations which are abelian on the trefoil, and blow up on the splitting torus.

What has happened here is that most representations of the Whitehead link complement which carry a certain loop on a boundary torus to $\pm\mathrm{Id}$, must in fact carry it to $+\mathrm{Id}$. Further, there is a loop on the boundary of the trefoil complement which has to go to $-\mathrm{Id}$ for every irreducible representation. Gluing these curves together gives representations which cannot be reconciled in $\mathrm{SL}(2,\mathbb{C})$; however it shows that (with appropriate definitions) the surface U can be obtained via degenerations of representations into $\mathrm{PSL}(2,\mathbb{C})$.

We have already observed that $H^2(X;\mathbb{Z}_2)$ is nonzero and these representations do not lift. We outline the reason for this — the following elegant argument was shown to us by Andrew Casson. Let $\rho\colon G \to \mathrm{PSL}(2,\mathbb{C})$ be any representation. Then the natural projection $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$ can be pulled back to give a commutative diagram of groups and homomorphisms, where the vertical maps are central extensions:

$$\begin{array}{ccc} \mathbb{Z}_2 & & \mathbb{Z}_2 \\ \downarrow & & \downarrow \\ G' & \xrightarrow{\rho'} & \mathrm{SL}(2,\mathbb{C}) \\ \downarrow & & \downarrow \\ G & \xrightarrow{\rho} & \mathrm{PSL}(2,\mathbb{C}) \end{array} .$$

Now the action of G on $\mathbb{Z}_2$ is trivial, and so the possibilities for groups G' correspond to elements of $H^2(G;\mathbb{Z}_2)$. If G is the fundamental group of a Haken 3-manifold M, then $H^2(M;\mathbb{Z}_2) \cong H^2(G;\mathbb{Z}_2)$. By Poincaré duality one sees that if the boundary of M is a torus and $H_1(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ then $H^2(M;\mathbb{Z}_2) \cong 0$. Thus $G' \cong \mathbb{Z}_2$, and so $\rho'|G$ is a lift of ρ to $\mathrm{SL}(2,\mathbb{C})$.

In particular, this shows that for knot complements in S^3 representations into $\mathrm{PSL}(2,\mathbb{C})$ always lift (compare **[C]**). Thus an incompressible surface in a knot complement is detected by $\mathrm{SL}(2,\mathbb{C})$ if and only if it is detected by $\mathrm{PSL}(2,\mathbb{C})$.

Concluding Remarks. This paper grew out of an attempt to show that every knot has an irreducible representation into $\mathrm{SL}(2,\mathbb{C})$, the idea being that one would try to patch together representations across incompressible tori. In the notation established above, this means that one needs to understand the image $i_0(E(\pi_1(W)))$. This involves two parts; one first needs to show that $i_0(E(\pi_1(W)))$ has complex dimension two, then to understand the forbidden curves. Both of these problems appear to be hard. One means of attack is given by:

Conjecture. *Suppose that M is a compact hyperbolic 3-manifold with boundary a torus and that the interior of M admits a complete hyperbolic structure ρ. Suppose that α is any non-trivial element of the fundamental group of M. Is it true that $\mathrm{tr}(\rho\alpha)$ is not constant on the component of the representation space containing ρ?*

An affirmative answer to this conjecture would mean that the projection of the character variety of a 2-component hyperbolic link N to the character variety of one of its boundary components T say always has complex dimension 2. The reason is that by **[T]** the component of the character variety of N containing the complete representation has complex dimension 2. Suppose that the restriction of this character variety to T has dimension 1, then there is a hyperbolic Dehn filling of T using a representation ρ of N. This gives a 3-manifold M with a single torus boundary component T' say, and since ρ may be varied keeping $\rho|T$ fixed, it follows that the trace of the core of the Dehn filling does not vary as ρ is varied near the complete representation of N.

Finally, we mention that although the most useful notion for a slope is that it be detected, there is a weaker notion: One may say that a slope is ***weakly detected*** if it is the boundary of some detected surface. The result of **[CS]** show that detected slopes are weakly detected, however the annulus in a connected sum shows that the converse may be false.

2. Appendix: Some calculations

With respect to the meridians shown below (Fig. 4), the fundamental group of the complement of the Whitehead link is presented as

$$< \mu_i, \mu_0 \mid \mu_i^{-1}\mu_0\mu_i\mu_0^{-1}\mu_i\mu_0\mu_i^{-1}\mu_0^{-1} = \mu_i\mu_0^{-1}\mu_i^{-1}\mu_0\mu_i^{-1}\mu_0^{-1}\mu_i\mu_0 > .$$

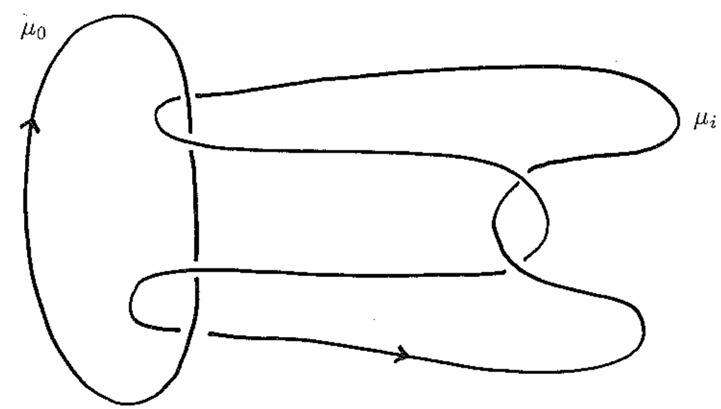

Figure 4

We are interested in representations up to conjugacy, so one easily finds that it is sufficient to consider representations

$$\mu_i \mapsto \begin{pmatrix} m_i & 1 \\ 0 & 1/m_i \end{pmatrix} \quad \text{and} \quad \mu_o \mapsto \begin{pmatrix} m_o & 0 \\ t & 1/m_o \end{pmatrix}.$$

Then the condition that we obtain a representation is the single polynomial

$$\begin{aligned}F(m_i,m_0,t) &= -(m_im_0)+m_i^3m_0+m_im_0^3-m_i^3m_0^3+\\ &+(m_i^2-m_i^4+m_0^2-4m_i^2m_0^2+m_i^4m_0^2-m_0^4+m_i^2m_0^4)t+\\ &+(-(m_im_0)+2m_i^3m_0+2m_im_0^3-m_i^3m_0^3)t^2-m_i^2m_0^2t^3=0.\end{aligned}$$

The theory of **[CS]** usually uses traces and the character variety, but this involves a loss of information in the sign of the slopes when considering the Newton polygon.

Briefly, one can form an eigenvalue variety as follows. Let $X(\pi_1(W))$ be a component of the affine algebraic set of characters. Then there is a projection map $X(\pi_1(W)) \to X(\pi_1(\partial W))$, coming from the restriction map, the target being some subvariety of $X(\pi_1(T_0)) \times X(\pi_1(T_i))$.

Considering the relevant component of the representation variety $R(\pi_1(W))$, we may form by projection an eigenvalue variety for the boundary $E(\pi_1(\partial W))$. Then there is a map $E(\pi_1(\partial W)) \to X(\pi_1(\partial W))$ which is generically two to one. Then the eigenvalue variety $E(\pi_1(W))$ comes from forming the pull back in the diagram

$$\begin{array}{ccc} & & E(\pi_1(\partial W)) \\ & & \downarrow \\ X(\pi_1(W)) & \longrightarrow & X(\pi_1(\partial W)) \end{array}.$$

We are interested in finding what eigenvalues are possible on the boundary. To do this one uses elimination theory **[M]** for which we briefly recall some details. Given polynomials

$$f(X_1,\ldots,X_n,Y)=\sum_{i=0}^{n} p_{n-i}(X_1,\cdots,X_n)Y^i$$

and

$$g(X_1,\ldots,X_n,Y)=\sum_{i=0}^{m} q_{m-i}(X_1,\cdots,X_n)Y^i,$$

we can form their resultant $R(X_1,\ldots,X_n)$, a polynomial having the following property. If $\alpha_1,\ldots,\alpha_n$ are complex numbers which satisfy $R(\alpha_1,\ldots,\alpha_n)=0$, then either:

(i) the polynomials $f(\alpha_1,\ldots,\alpha_n,Y)$ and $g(\alpha_1,\ldots,\alpha_n,Y)$ have a common root, or

(ii) $p_0(\alpha_1,\ldots,\alpha_n)=q_0(\alpha_1,\ldots,\alpha_n)=0$.

In our context we use this in the following way. One computes that the longitude on the outside torus is the word

$$\mu_0^{-1}\mu_i^{-1}\mu_0\mu_i\mu_0^{-1}\mu_i\mu_0\mu_i^{-1},$$

and one computes using the above generators a rational function $Q(m_0,m_i,t)$ which is the $(1,1)$ entry of this matrix; that is to say, if we set ℓ_0 to be the eigenvalue of the longitude on the outer torus, we have $\ell_0=Q(m_0,m_i,t)$, which may be re-written as a polynomial function $g(m_0,m_i,t,\ell_0)=0$. We then eliminate the variable t between this polynomial and $F(m_i,m_0,t)$.

This yields:

$$\begin{aligned}O(m_i,m_0,\ell_0) &= \ell_0(-1+m_0)(1+m_0)(1+\ell_0m_0^2)+\\ &+m_i^4\ell_0(-1+mo)(1+m_0)(1+\ell_0m_0^2)+\\ &+m_i^2(1+\ell_0^2+2\ell_0m_0^2-2\ell_0^2m_0^2-\ell_0m_0^4-\ell_0^3m_0^4).\end{aligned}$$

In general, elimination theory involves loss of information in the affine context because of possibility (ii). However, in our case the highest power of t occurring in F is a cube and its coefficient is $m_i^2m_0^2$ so that we see that given any triple of nonzero complex numbers m_i, m_0, ℓ_0 which satisfy $O(m_i,m_0,\ell_0)=0$, (i) applies and we may find a representation of $\pi_1(W)$ which gives this eigenvalue triple on the boundary.

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