The Area of Convex Projective Surfaces and Fock-Goncharov Coordinates

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Abstract. The area of a convex projective surface of genus $g \geq 2$ is at least $(g - 1)\pi^2/2 + \|\tau\|^2/8$ where $\tau = (\log t_i)$ is the vector of triangle invariants of Bonahon-Dreyer and $t_i$ are the Fock-Goncharov triangle coordinates.

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A convex projective surface is $F = \Omega/\Gamma$ where $\Omega \subset \mathbb{R}P^2$ is the interior of a compact convex set disjoint from some projective line, and $\Gamma \subset \text{PGL}(3, \mathbb{R})$ is a discrete subgroup that preserves $\Omega$ and acts freely on it. An example is a hyperbolic surface. Let $\mathbb{R}P(S)$ be the space of marked convex real-projective structures on a closed orientable surface $S$. Goldman and Choi [8] showed that this can by identified with the Hitchin component of $\text{Hom}(\pi_1 S, \text{PGL}(3, \mathbb{R}))/\text{PGL}(3, \mathbb{R})$. Bonahon and Dreyer [7] showed:

Theorem 0.1. Suppose $S$ is a closed orientable surface of genus $g \geq 2$. There are real analytic maps, the triangle invariant $\tau : \mathbb{R}P(S) \to \mathbb{R}^{4g-4}$, and the shear invariant $\sigma : \mathbb{R}P(S) \to \mathbb{R}^{12g-10}$, so that $\beta = (\tau, \sigma) : \mathbb{R}P(S) \to \mathbb{R}^{16g-14}$ is a real-analytic parameterization. The image of $\beta$ is the an open cone defined by 2 linear inequalities and 2 linear equalities. The image of $\tau$ is open in a codimension-1 subspace.

The Hilbert metric $d_\Omega$ on a bounded convex set $\Omega \subset \mathbb{R}^2$ is a Finsler metric normalized so that when $\Omega$ is the interior of the unit disc the Hilbert metric is the hyperbolic metric (curvature $-1$). A Finsler metric determines a measure called $p$-area or just area that is the largest measure in the same measure class as Lebesgue so that the area of an infinitesimal parallelogram is at most the product of the side lengths, see Definition [4]. In general this is not Hausdorff measure [4] but coincides with it for Riemannian metrics. For example the unit ball in the sup-norm is a square of side-length 2 which has $p$-area equal to the Euclidean area 4, and in the taxicab norm the unit
ball has p-area 2 but the Hausdorff measure of the unit ball of a norm is always π. When applied to the Hilbert metric we call p-area the Hilbert area or just area. In the positive quadrant the Hilbert area form is \(dxdy/(4xy)\), see Lemma 24.

An ideal triangle in \(\Omega\) has a shape parameter \(t > 0\). This is one of the coordinates introduced by Fock and Goncharov [13]. The triangle invariant in the Bonahon-Dreyer theorem for this triangle is \(\tau = \log t\) which is a certain signed distance, see Figures 2 and 4. Our main result is:

**Theorem 0.2.** Suppose \(T\) is an ideal triangulation of a surface \(F\). The map \(\alpha_T : \mathbb{RP}(S) \to \mathbb{R}\) defined by

\[
\alpha_T(F) = (g - 1)\pi^2/2 + \|\tau\|^2/8
\]

satisfies \(\text{area}(F) \geq \alpha(F)\). Here \(\tau = (\tau_1, \cdots, \tau_{4g-4})\) are the components of \(\tau\) and \(\|\tau\|^2 = \sum \tau_i^2\).

This follows from:

**Proposition 0.3.** If \(T\) is an ideal triangle with shape parameter \(t\) in a properly convex domain \(\Omega\) then

\[
\text{area}_\Omega(T) \geq (\pi^2 + (\log t)^2)/8
\]

with equality iff \(\Omega\) is the interior of a triangle.

The area of a compact hyperbolic surface \(F\) is \(2\pi|\chi(F)|\) and in particular is bounded below. A corollary of the above gives a lower bound on the area of a convex projective surface by using an ideal triangulation of the surface.

**Theorem 0.4.** If \(F\) is a compact, properly convex projective surface then

\[
\text{area}(F) \geq (\pi/2)^2 \cdot |\chi(F)|.
\]

It follows that if \(Q\) is a compact 2-orbifold with \(\chi^{orb}(Q) < 0\) then every convex projective structure on \(Q\) has area at least \((\pi/2)^2 \cdot |\chi^{orb}(Q)|\). In particular this area is bounded below by \(\pi^2/168\). The first author has given lower bounds on the volumes of hyperbolic orbifolds [4]; and with Guofang Wei see also [2] and for complex hyperbolic orbifolds [3].

With the above normalization the Hilbert metric on the interior of the unit disc equals the hyperbolic metric. The above lower bound is \(\pi/8 \approx 38\%\) of the hyperbolic area. In the remainder of this paper we normalize the Hilbert metric so that it is twice that above. This means the areas calculated in the rest of the paper should be divided by 4 to give the results announced.

This result extends to complete convex projective structures 2-orbifolds and to surfaces with cusps. Theorem 0.2 suggests various questions. For example there is a graph \(\Gamma\) with a vertex for each ideal triangulation \(T\) of \(F\) and edges that correspond to edge flips. Given a strictly convex structure on \(F\) what are the properties of the function defined on the vertices of \(\Gamma\) by \(T \mapsto \alpha_T(F)\)? Is the maximum attained? Is there a uniform upper bound on the difference between the maximum and the area of \(F\)?
It follows from the Margulis lemma for convex projective orbifolds [10], [11] that there is a lower bound on the Hilbert volume of a strictly convex projective $n$-orbifold. What are explicit lower bounds?

The proposition is proved by using the following observation. An ideal triangle $T$ in a domain $\Omega$ meets $\partial \Omega$ at its three vertices. There is a triangle $\Delta$ that contains $\Omega$ and is tangent to $\Omega$ at these three points, see Figure 2. The Hilbert metric given by $\Delta$ is smaller than that given by $\Omega$. Thus the area of $T$ in the Hilbert metric on $\Omega$ is bigger than its area using the Hilbert metric from $\Delta$. We explicitly calculate the latter. The theorem follows by using ideal triangulations.

After the first version of this paper was written the authors learned of [9]. In that paper the authors show that the (Hausdorff) area of an ideal triangle is bounded below, and also that it is not bounded above. Their lower bound is different because they use a different definition of area. However, as Marquis remarks in [15], although there are different notions of area for Finsler metrics, because of the Benzécri compactness theorem [6], there is a universal bound on the ratios of different area forms for reasonable choices. The point of this paper is the relation to Fock-Goncharov coordinates.

1. Length and Area in Hilbert geometry

The cross-ratio of four distinct points $y_1, y_2, y_3, y_4 \in \mathbb{R}$ is

$$cr(y_1, y_2, y_3, y_4) = \frac{(y_1 - y_3)(y_2 - y_4)}{(y_1 - y_2)(y_3 - y_4)}$$

Using the embedding $\mathbb{R} \subset \mathbb{R}P^1$ given by $y/x \mapsto [x : y]$ cross-ratio extends to a continuous map $cr : X \to \mathbb{R}P^1$ where $X \subset (\mathbb{R}P^1)^4$ is the subset of quadruples of points at least 3 of which are distinct:

$$cr([x_1 : y_1], [x_2 : y_2], [x_3 : y_3], [x_4 : y_4]) = \left[\frac{(x_2y_1 - x_1y_2)(x_4y_3 - x_3y_4) : (x_3y_1 - x_1y_3)(x_4y_2 - x_2y_4)}{(x_2y_1 - x_1y_2)(x_4y_3 - x_3y_4)}\right].$$

Suppose $\Omega \subset \mathbb{R}^n \subset \mathbb{R}P^n$ is an open convex set that contains no affine line. Given $b, c \in \Omega$ there is a projective line $\ell$ in $\mathbb{R}P^n$ that contains them. This line meets $\partial \Omega \subset \mathbb{R}P^n$ in two distinct points $a, d$. Label these points so $a, b, c, d$ are in linear order along $\ell \cap \mathbb{R}^n$. Since $\Omega$ contains no affine line $a \neq d$.

The Hilbert metric on $\Omega$ is

$$d_\Omega(b, c) = |\log cr(a, b, c, d)|$$

Some authors use $(1/2)$ of this so that when $\Omega$ is the unit ball then $d_\Omega$ has curvature $-1$. This is a Finsler metric and

$$d_\Omega(x, x + dx) = \left(\frac{1}{|x - a|} + \frac{1}{|x - b|}\right) dx$$

In particular if $\Omega = (0, \infty) \subset \mathbb{R}^1$ then $a = 0, b = \infty$ and $d_\Omega(x, x + dx) = dx/x$.

The literature contains many distinct definitions of area for Finsler metrics [4], [5]. These depend on how area is defined for a normed vector space.
Area is a Borel measure on $V$ that is preserved by translation (i.e., a Haar measure) so it is some multiple of Lebesgue measure. We will adopt the following definition which is particularly well suited for our purposes.

**Definition 1.1.** Suppose $(V, \| \cdot \|)$ is a normed 2-dimensional real vector space. Choose an inner product on $V$ and let $\lambda$ be the resulting Lebesgue measure. Set $K = \sup \lambda(\{a\alpha + b\beta : 0 \leq \alpha, \beta \leq 1\})$ where the supremum is over all $a, b \in V$ with $\|a\|, \|b\| \leq 1$. Then define the parallelogram measure or $p$-area $\omega_{\| \cdot \|}$ on $V$ by $\omega_{\| \cdot \|} = K^{-1} \cdot \lambda$.

It is easy to check the definition is independent of the choice of inner product. Having done this we dispense with the inner product, and refer to the norm of a vector as its length. The definition is equivalent to declaring that the maximum area of a parallelogram with sides of unit length is 1. If the norm is given by an inner product then parallelogram measure coincides with the usual area. This definition generalizes to $n$ dimensions using parallelopipeds spanned by $n$ vectors of norm one.

A rectangle in $(V, \| \cdot \|)$ is defined to be any parallelogram with side lengths $x$ and $y$ and area $xy$. Such parallelograms always exist. This enables the standard construction of Lebesgue measure in the plane, starting from an inner product, to be extended to an arbitrary norm on the plane.

Parallelogram measure is an increasing function of the metric in the sense that if $\| \cdot \|$ and $\| \cdot \|'$ are two norms on $V$ with $\| \cdot \| \leq \| \cdot \|'$ then $\omega_{\| \cdot \|} \leq \omega_{\| \cdot \|'}$. In particular if $\alpha > 0$ then $\omega_{\alpha \| \cdot \|} = \alpha^2 \omega_{\| \cdot \|}$.

A Finsler surface is a pair $(S, ds_x)$ where $S$ is a smooth surface and $ds_x$ is a norm on $T_x S$ for each $x \in S$. The $p$-area form on $S$ is the $p$-area form for $ds_x$ on $T_x S$. For a properly convex projective surface $S$ the resulting area form $\omega_S$ is called the Hilbert area form, and the Hilbert area of $S$ is $\int_S \omega_S$. If $\Omega' \subset \Omega$ are properly convex then $d\Omega \leq d\Omega'$ on $\Omega'$ and $\mu_\Omega \leq \mu_{\Omega'}$.

### 2. Hex geometry

A reference for this section is [12]. Let $u_0, u_1, u_2 \in \mathbb{R}^2$ be unit vectors with respect to the standard inner product such that $u_0 + u_1 + u_2 = 0$. We will use $u_0 = (1, 0)$ and $u_1 = (-1/2, \sqrt{3}/2)$ then $u_2 = -u_0 - u_1$. The convex hull of the vectors $\{\pm u_0, \pm u_1, \pm u_2\}$ is a regular Hexagon $H$.

**Definition 2.1.** The Hex plane $(\mathcal{H}, d_\mathcal{H})$ is the metric space obtained from the norm on $\mathbb{R}^2$ with unit ball $H$.

For the Hex plane $p$-area is different to Busemann volume, used for example in [15], or Holmes-Thompson used in [17]. On a normed plane all these measures are multiples of Haar measure, and so they are multiples of each other. We will describe some properties of $p$-area for the Hex plane which suggest this is the right definition to use.

**Lemma 2.2.** Let $\omega_\mathcal{H}$ denote the area form on the Hex plane and $\lambda$ be Lebesgue measure with respect to the standard inner product. Then $\omega_\mathcal{H} = (2/\sqrt{3})\lambda$.
Proof. Let $a, b$ be unit vectors in the Hex norm. Then they lie on the boundary of the regular unit Hexagon $H$ center at the origin and determine a parallelogram $P(a, b)$ with vertices $\{0, a, b, a + b\}$. Suppose $a$ lies on the edge $e$ of $H$. Then the area of $P(a, b)$ is maximized by taking $b$ to be a vertex of $H$ that that is not an endpoint of $\pm e$. The Euclidean area of $P(a, b)$ is then $\sqrt{3}/2$. □

Our choice of normalization of area has the following consequences in the Hex plane. A Hex circle of radius $r$ is a Euclidean regular Hexagon so that the (Euclidean=Hex) distance from the center to a vertex is $r$. The circumference of this circle is $6r$ and the p-area is $3r^2$.

The positive quadrant is $Q = \{(x, y) : x, y > 0 \}$. A triangle in $\mathbb{R}P^2$ is a compact convex subset, $\Delta$, bounded by 3 segments of projective lines. There is a projective transformation taking the interior of $\Delta$ to $\{[x : y : 1] : x, y > 0 \}$ which may be identified with $Q$. Thus the Hilbert metric on the interior of $\Delta$ is isometric to $(Q, d_Q)$.

Lemma 2.3 (Proposition 7 in [12]). The Hilbert metric on $Q$ is isometric to the Hex plane $(\mathcal{H}, d_\mathcal{H})$.

Proof. There is an isometry $A : (\mathbb{R}^2, \| \cdot \|_{Hex}) \rightarrow (Q, d_Q)$ given by

$$A(u, v) = \left(e^{u + \frac{1}{\sqrt{3}} v}, e^{\frac{2}{\sqrt{3}} v}\right) = (x, y).$$

This may be checked as follows. The map $A$ conjugates the action of $\mathbb{R}^2$ on itself by translations to the action of the positive diagonal group on $Q$. Thus it suffices to check $A$ is infinitesimally an isometry at the origin. □

If $\Omega \subset \mathbb{R}P^2$ is a properly convex domain the Hilbert area form $\omega_\Omega$ is the 2-form given by the Hilbert metric.
Lemma 2.4. The Hilbert area form on the positive quadrant $Q = \{(x, y) : x, y > 0\}$ is

$$\omega_Q = \frac{dx dy}{xy}$$

Proof. The isometry $A$ in Lemma 2.3 infinitesimally multiplies Lebesgue measure $\lambda$ on $\mathbb{R}^2$ by

$$J = \left| \begin{array}{cc}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{array} \right| = \left| \begin{array}{cc}
e^{u+\frac{1}{\sqrt{3}}v} & \frac{1}{\sqrt{3}}e^{u+\frac{1}{\sqrt{3}}v} \\
0 & \frac{2}{\sqrt{3}}e^{2\sqrt{3}v}
\end{array} \right| = \frac{2}{\sqrt{3}} \left( e^{u+\frac{1}{\sqrt{3}}v} \right) \left( e^{\frac{2}{\sqrt{3}}v} \right) = \frac{2}{\sqrt{3}}xy.$$

Since $\omega_H = (2/\sqrt{3})\lambda$ it follows that

$$\omega_Q = \frac{\omega_H}{2\sqrt{3}xy} = \frac{\lambda}{xy} = \frac{dx dy}{xy}.$$

Lemma 2.5. The p-area of every parallelogram in the Hex plane is base $\times$ height.

Proof. Affine maps of the plane multiply Euclidean area (and hence p-area) by the determinant of the linear part. Hence it suffices to prove the result in the special case of a parallelogram $P$ with vertices $0, v_0, v_1, v_0 + v_1$ for which the length of the base is $\|v_0\|_{Hex} = 1$ and the height is also one. Moreover a rotation through an angle of $\pi/3$ is an isometry of Hex. Thus we may assume $v_0$ is on the side of $H$ between $u_0$ and $u_0 + u_1$.

Refer to Figure 1. We transform $P$ to $P'$ to $P''$ and show that each of these parallelograms has the same p-area. A shear parallel to the base of $P$ preserves base, height and p-area. Shear $P$ parallel to $v_0$ sending $v_1$ to $u_1$ to give a parallelogram $P'$ with vertices $0, v_0, u_1, v_0 + u_1$. Now shear $P'$ parallel to the $u_1$-direction to get a parallelogram $P''$ with vertices $0, u_0, u_1, u_0 + u_1$. This shear preserves base and height because of the special properties of $H$. The area of $P''$ is 1.

3. Ideal triangles in Hex geometry

Suppose $\Omega \subset \mathbb{R}^2$ is an open properly convex set. If $T$ is a triangle with vertices in $\partial \Omega$ then $T \cap \Omega$ is called an ideal triangle in $\Omega$. It is proper if $T \cap \partial \Omega$ consists of only the vertices of $T$. We will only be concerned with proper ideal triangles, and will henceforth omit the term proper.

If $\Delta = \{[x_0v_0 + x_1v_1 + x_2v_2] : x_i > 0 \}$ is the interior of triangle in $\mathbb{R}^2$ there is an isometry $\phi : (\Delta, d_\Delta) \rightarrow (\mathcal{H}, d_\mathcal{H})$ given by

$$\phi[x_0v_0 + x_1v_1 + x_2v_2] = (\log x_0)u_0 + (\log x_1)u_1 + (\log x_2)u_2$$

Suppose $T$ is an ideal triangle in $\Delta$. We refer to $\phi(T)$ as an ideal triangle in the Hex plane. Then $T \subset \Delta$ together with an ordering of the vertices of $\Delta$ determines a shape parameter $t = t(T, \Delta) \in \mathbb{R}$ defined as follows, see Figure 2. If the vertices of $T$ are $[w_0 = v_0 + av_1], [w_1 = v_1 + bv_2], [w_2 = v_2 + cv_0]$
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\[ \begin{bmatrix} 1 : 0 : 0 \\ 0 : 1 : 0 \end{bmatrix} \]

\[ \begin{bmatrix} 1 : a : 0 \\ 0 : 1 : b \end{bmatrix} \]

\[ \begin{bmatrix} 0 : 0 : 1 \\ 1 : 1 : 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 : 1 : t \\ 0 : 1 : 1 \end{bmatrix} \]

\[ \begin{bmatrix} 1 : 0 : 1 \\ 1 : 0 : 0 \end{bmatrix} \]

\[ \begin{bmatrix} c : 0 : 1 \\ 0 : 1 : 1 \end{bmatrix} \]

**Figure 2.** The shape parameter \( t > 0 \) and triangle invariant \( \tau = \log t \)

with \( a, b, c > 0 \) then \( t = abc \). This depends only on the cyclic ordering of the vertices. Changing this ordering replaces \( t \) by \( 1/t \). Observe that when \( a = b = 1 \) then \( |\log t| \) is the Hilbert distance in \((v_1, v_2)\) between \([w_2]\) and the midpoint \([v_1 + v_2]\).

The group \( PGL(\Omega) \) is the subgroup of \( PGL(3, \mathbb{R}) \) which preserves \( \Omega \). Using the basis \( v_0, v_1, v_2 \) of \( \mathbb{R}^3 \) the identity component \( PGL_0(\Delta) \) consists of positive diagonal matrices. This group acts transitively on the interior of \( \Delta \). There is a unique element \( \tau \in PGL_0(\Delta) \) which takes two of the vertices of \( T \) to \([u_1 + u_2], [u_1 + u_3]\). The remaining vertex is taken to \([u_2 + tu_3]\), see Figure 2. The regular ideal triangle is given by \( t = 1 \). It has maximal isometry group: dihedral of order 6.

**Proposition 3.1.** Isometry classes of ideal Hex triangle are in 1-1 correspondence with shape parameters \( t \in [1, \infty) \).

**Lemma 3.2.** The Hilbert area of an ideal triangle in the Hex plane with shape parameter \( t \) is

\[ B(t) = \int_1^\infty \frac{1}{s} \log \left( \frac{(st + 1)(s + t)}{t(s - 1)^2} \right) ds \]

**Proof.** By Lemma 2.3 the ideal triangle is isometric to an ideal triangle \( T \) in \( Q \). By means of a projective transformation preserving \( Q \) we can arrange that one side of \( T \) is given by \( x + y = 1 \) and the other two sides are then the parallel rays in \( Q \) given by \( y - 1 = tx \) and \( y = t(x - 1) \). Refer to Figure 3. It is easy to check that \( t \) is the shape parameter.

For \( s \geq 1 \) define \( \alpha(s), \beta(s) \) to be the points of intersection of the line \( x + y = s \) with the sides \( y - 1 = tx \) and \( y = tx + 1 \) of \( T \). Let \( \gamma(s) \) be the line segment with these endpoints. For \( s \geq 1 \) these lines foliate \( T \). For \( s > 1 \) define

\[ \ell(s) = d_Q(\alpha(s), \beta(s)) \]

to be the Hilbert length of \( \gamma(s) \). The distance between \( \gamma(s) \) and \( \gamma(s + ds) \) is \( ds/s \). This is easily seen by projecting onto the \( x \)-axis along the direction
\[ x + y = 0. \] It follows from Lemma 2.5 that the Hilbert area of the infinitesimal parallelogram shown is \( \ell(s) ds/s \), thus

\[
\text{area}_\Delta(T) = \int_1^\infty \frac{\ell(s)}{s} \, ds
\]

Next we compute

\[
\ell(s) = \log \left( \frac{(s + t)(st + 1)}{t(s - 1)^2} \right)
\]

This may be done as follows. Projection of one line onto another preserves Hilbert distance. Let \( \pi \) be vertical projection of the line segment \( x + y = s \) in \( Q \) onto the \( x \)-axis. The image of this segment is \([0, s]\). Then

\[
\pi(\alpha(s)) = \frac{s - 1}{t + 1}, \quad \pi(\beta(s)) = \frac{s + t}{t + 1}
\]

Then \( \ell(s) = d_Q(\alpha(s), \beta(s)) = d_{[0,s]}(\alpha(s), \beta(s)) = |\log CR(0,\alpha(s), \beta(s), s)| \).

□

Next we calculate this integral. This implies Proposition 0.3.

**Lemma 3.3.**

\[ B(t) = \frac{\pi^2 + (\log t)^2}{2} \]

**Proof.**

\[
\frac{dB}{dt} = \frac{d}{dt} \left( \int_1^\infty \frac{1}{s} \log \left( \frac{(st + 1)(s + t)}{t(s - 1)^2} \right) \, ds \right)
\]

Taking the derivative inside the integral gives

\[
\int_1^\infty \frac{1}{s} \frac{d}{dt} \left( \log \left( \frac{(st + 1)(s + t)}{t(s - 1)^2} \right) \right) \, ds = \int_1^\infty \frac{1}{s} \left( \frac{s}{1 + st} + \frac{1}{s + t} - \frac{1}{t} \right) \, ds
\]

\[
= \int_1^\infty \left( \frac{1}{1 + st} - \frac{1}{t(s + t)} \right) \, ds
\]

\[
= \left[ \frac{1}{t} \left( \log(1 + st) - \log(s + t) \right) \right]_{s=1}^{s=\infty}
\]

\[
= \left[ t^{-1} \log \left( \frac{1 + st}{s + t} \right) \right]_{s=1}^{s=\infty}
\]

\[
= t^{-1} \log t
\]

Thus

\[
\frac{dB}{dt} = t^{-1} \log t
\]

Integrating gives \( B(t) = (1/2)(\log t)^2 + C \). The next lemma shows \( C = \pi^2/2 \).

□

**Lemma 3.4.**

\[ B(1) = \frac{\pi^2}{2} \]
Proof. 

\[ B(1) = \int_1^{\infty} \frac{1}{s} \log \left( \frac{(s+1)^2}{(s-1)^2} \right) \, ds = \int_1^{\infty} \frac{2}{s} \log \left( \frac{s+1}{s-1} \right) \, ds \]

Set \( w = (s + 1)/(s - 1) \) then

\[ s = \frac{w+1}{w-1} = 1 + \frac{2}{w-1} \quad \text{and} \quad \frac{ds}{dw} = -2(w-1)^{-2} \]

hence

\[ B(1) = \int_1^{\infty} \frac{2}{s} \log \left( \frac{s+1}{s-1} \right) \, ds = \int_{\infty}^{1} 2 \left( \frac{w-1}{w+1} \right) \log w (-2)(w-1)^{-2} \, dw \]

\[ = \int_{1}^{\infty} \frac{4 \log w}{w^2 - 1} \, dw \]

Set \( w = e^x \) this becomes

\[ B(1) = \int_0^{\infty} \frac{4x \, e^x \, dx}{e^{2x} - 1} = \int_0^{\infty} \frac{4x \, dx}{e^x - e^{-x}} \]

The integrand is even so

\[ B(1) = 2 \int_0^{\infty} \frac{2x \, dx}{e^x - e^{-x}} \]

Define

\[ g(z) = \frac{i}{4\pi} z(z - 2\pi i), \]

then

\[ g(z) - g(z + 2\pi i) = z. \]

Hence

\[ B(1) = 2 \int_{-\infty}^{\infty} \frac{g(x) - g(x + 2\pi i)}{e^x - e^{-x}} \, dx. \]
We claim this equals the limit of the contour integrals

\[ B(1) = \lim_{R \to \infty} \oint_{\Gamma_R} \frac{2g(z)}{e^z - e^{-z}} \, dz. \]

Where \( \Gamma_R \) is the rectangle

\[ (-R, R) \times \{0, 2\pi i\} \cup \{\pm R\} \times [0, 2\pi i] \]

oriented counterclockwise. Observe that on the vertical sides \((\pm R) \times [0, 2\pi i]\) the integrand goes to 0 as \( R \to \infty \). The integral along the horizontal sides \((-R, R) \times \{0, 2\pi i\}\) with the orientation specified is

\[ 2 \int_{-R}^{R} g(z) - g(z + 2\pi i) \frac{e^z - e^{-z}}{z} \, dz. \]

This proves the claim. To evaluate the contour integral we observe the only singularity inside \( \Gamma_R \) of

\[ \frac{2g(z)}{e^z - e^{-z}} \]

is a simple pole at \( z = i\pi \). Now

\[ 2g(i\pi) = 2 \frac{i(i\pi)(-i\pi)}{4\pi} = \frac{i\pi}{2}. \]

Also the denominator is \( e^z - e^{-z} = 2z + \cdots \) has residue \(-1/2\) at \( z = i\pi \) because

\[
\text{residue} = \lim_{z \to i\pi} \frac{z - i\pi}{e^z - e^{-z}} \\
= \lim_{w \to 0} \frac{w}{e^{w+i\pi} - e^{w-i\pi}} \\
= \lim_{w \to 0} \frac{w}{e^{-i\pi}(e^w - e^{-w})} \\
= -\lim_{w \to 0} \frac{w}{2w + \cdots} \\
= -\frac{1}{2}.
\]
Thus the residue of the integrand at $i\pi$ is $(-1/2)(i\pi/2) = -i\pi/4$. Cauchy’s theorem gives the contour integral is

$$(2\pi i)(-i\pi/4) = \pi^2/2.$$ 

□

4. Ideal triangulations

Ideal triangulations of surfaces were introduced by Thurston for hyperbolic surfaces, see [14],[16]. The extension to properly convex surfaces is routine.

**Definition 4.1.** An ideal triangulation of a convex projective surface $F = \Omega/\Gamma$ is a decomposition of $F$ into closed subsets with disjoint interiors called ideal triangles such that each component of the preimage of an ideal triangle in $F$ is an ideal triangle in $\Omega$.

**Proposition 4.2.** If $F$ is a closed convex projective surface then $F$ has an ideal triangulation. The number of ideal triangles is $2|\chi(F)|$.

Suppose $F = \Omega/\Gamma$ is a compact, strictly convex projective surface with $\chi(F) < 0$. Then $\Omega \subset \mathbb{R}P^2$ has a unique tangent line at each point of $\partial\Omega$ by [6]. If $T \subset \Omega$ is an ideal triangle with vertices $p, q, r \in \partial\Omega$. The tangent lines at $p, q, r$ contain the sides of a triangle $\Delta$ which contains $\Omega$. Following Fock and Goncharov [13] we define the shape $t = t(T, \Omega) \in [1, \infty)$ of $T$ in $\Omega$ to be the shape $t(T, \Delta)$ of $T$ in $\Delta$ previously defined.

For a properly convex set $\Omega$ we denote the Hilbert area form on $\Omega$ by $\omega_\Omega$. This pushes down to a 2-form $\omega_F$ on $F$. We denote the Hilbert area of a
measurable subset $X \subset F$ by

$$area_F(X) = \int_X \omega_F$$

Since $\Omega \subset \Delta$ it follows that $d_\Delta \leq d_\Omega$ and $\omega_\Delta \leq \omega_\Omega$. It follows that if $T \subset F$ is an ideal triangle with shape parameter $t$ then

$$area_F(T) \geq B(t) \geq \pi^2/2$$

Since $F$ contains $2|\chi(F)|$ ideal triangles with disjoint interiors Theorem 0.2 follows from Propositions 4.2 and 0.3.

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