MATH 118A, FALL 2014, PROBLEM SET 2
DUE WEDNESDAY, OCTOBER 15

1. *Fun with supremums*

Let $S \subset \mathbb{R}$ and $T \subset \mathbb{R}$ be two nonempty subsets of $\mathbb{R}$, each bounded from above.

(a) Suppose $s \leq x$ for all $s \in S$. Show that $\sup(S) \leq x$.

(b) Let $x \in \mathbb{R}$. Define $x + S = \{x + s : s \in S\}$. Show that $\sup(x + S) = x + \sup(S)$.

(c) Define $S + T = \{s + t : s \in S,\ t \in T\}$. Show that $\sup(S + T) = \sup(S) + \sup(T)$.

Hint: If you're stuck on how to prove these equalities in (b) and (c), here's a tip: First prove LHS $\leq$ RHS and then prove LHS $\geq$ RHS. Part (a) is useful.

2. *Roots of polynomials of odd degree*
In this problem, we'll prove that the polynomial $p(x) = x^5 - 3x + 1 = 0$ has at least one root in the real numbers. That is, there exists $t \in \mathbb{R}$ such that $p(t) = 0$.

Challenge (not for credit): The techniques of this problem can be extended to prove the result for any odd degree polynomial. Can you prove this much more general claim?

(a) Prove that $p(x) < 0$ for $x < -100$ and that $p(x) > 0$ for $x > 100$.

Remark: Hence the set $S = \{x : p(x) \leq 0\}$ is nonempty and is bounded from above.

(b) Show that if $|x| \leq 1$ and $|h| < 1$, then $|p(x + h) - p(x)| < 100|h|$.

Hint: Expand out $p(x + h) = (x + h)^5 - 3(x + h) + 1$. Note that $|h| \geq |h^2|$ etc.

(c) Show that if $|x| > 1$ and $|h| < 1$, then $|p(x + h) - p(x)| < 100|x|^4|h|$.

Hint: Here note that $|x^4| \geq |x^3|$ etc.

(d) Let $t = \sup\{x : p(x) \leq 0\}$. Prove that $p(t) = 0$.

Hint: Prove this by contradiction. Parts (b) and (c) will help. Part (a) was just to show this supremum exists.

3. [Rudin 1.6] *Exponentiation by a real number*

Fix a real number $b > 1$.

(a) Let m, n, p, $q \in \mathbb{Z}$ with n, $q > 0$ and with $m/n = p/q$. We call this $r = m/n = p/q$. Prove that

$$(b^m)^{1/n} = (b^p)^{1/q}$$

Hence for $r \in \mathbb{Q}$, we define $b^r = (b^m)^{1/n}$ when $r = m/n$, and this is well-defined (i.e. does not depend on the in what terms we expressed r as a fraction).

Note: Here $b^{1/a}$ is defined as the unique positive real number c such that $c^a = b$. You may assume that c exists and is unique.

Note: You may use that $(b^a)^b = b^{ab}$ when a and b are integers, but for this problem you may certainly not assume this when they are not integers, since we are defining what a rational power means, and thus may not assume it satisfies a property unless we prove it ourselves.

(b) Prove that $b^{r+s} = b^r b^s$ for $r, s \in \mathbb{Q}$

(c) Prove that, for $r \in \mathbb{Q}$, we have

$$b^r = \sup\{b^s : s \in \mathbb{Q}, s \leq r\}$$

Note: This supremum, like all others to come in this course, is taken in the reals. (Also, b^s for s rational is real but need not be rational.)

Now we define b^x for x real by the same expression: $b^x = \sup\{b^s : s \in \mathbb{Q}, s < x\}$.

(d) Prove that $b^{x+y} = b^x b^y$ for $x, y \in \mathbb{R}$.