COUNTING CONJUGACY CLASSES OF FULLY IRREDUCIBLES: DOUBLE EXPONENTIAL GROWTH

ILYA KAPOVICH, CATHERINE PFAFF

ABSTRACT. Inspired by results of Eskin and Mirzakhani [EM11] counting closed geodesics of length $\le L$ in the moduli space of a closed surface Σ_g of genus $g \ge 2$, we consider a similar question in the $Out(F_r)$ setting. Let $h = 6g - 6$. The Eskin-Mirzakhani result, giving the asymptotics of $\frac{e^{hL}}{hL}$, can be equivalently stated in terms of counting the number of $MCG(\Sigma_g)$-conjugacy classes of pseudo-Anosovs $\varphi \in MCG(\Sigma_g)$ with dilatation $\lambda(\varphi)$ satisfying $\log\lambda(\varphi) \le L$. For $L \ge 0$ let $\mathfrak{N}_r(L)$ denote the number of $Out(F_r)$-conjugacy classes of fully irreducibles $\varphi \in Out(F_r)$ with dilatation $\lambda(\varphi)$ satisfying $\log\lambda(\varphi) \le L$. We prove for $r \ge 3$ that as $L \to \infty$, the number $\mathfrak{N}_r(L)$ has double exponential (in L) lower and upper bounds. These bounds reveal behavior not present in classic hyperbolic dynamical systems.

1. INTRODUCTION

The theme of counting closed geodesics plays an important role in geometry and dynamics, and primarily dates back to the seminal work of Margulis in the 1960s-1970s [Mar69, Mar70]. Margulis considered the situation where M is a closed Riemannian manifold of curvature ≤ -1, and proved that if $\mathfrak{N}(L)$ is the number of closed geodesics in M of length $\le L$, then

$$\mathfrak{N}(L) \sim \frac{e^{hL}}{hL},$$

where h is the topological entropy of the geodesic flow on M (equivalently the volume entropy of M). Here $\sim$ means that the ratio of two functions converges to 1 as $L \to \infty$. This result can also be interpreted as counting the number of conjugacy classes $[\gamma]$ of elements $\gamma \in \pi_1(M)$ with translation length $\le L$ in $\widetilde{M}$. There were many generalizations of Margulis' result to other contexts, including manifolds with cusps, manifolds of nonpositive curvature, orbifolds, etc. The proofs of all these generalizations, as well as the original proof of Margulis, exploit the properties of the geodesic flow on the underlying structure and ultimately rely on some form of coding by symbolic dynamical systems. See the book of Buser [Bus92] for a detailed discussion of this subject, including history and additional references.

While the moduli space of a surface is not a manifold, recent important work of Eskin and Mirzakhani [EM11] provides an analog of Margulis' theorem in the moduli space setting. Namely, let Σ_g be a closed connected oriented surface of genus $g \ge 2$, let $\mathcal{T}(\Sigma_g)$ denote the Teichmüller space of Σ_g, endowed with the Teichmüller metric, and let $\mathcal{M}_g$ denote the moduli space, locally also equipped with the Teichmüller metric. Again, denote by $\mathfrak{N}_g(L)$ the number of closed Teichmüller geodesics in $\mathcal{M}_g$ of length $\le L$ and let $h = 6g - 6$. Eskin and Mirzakhani proved [EM11] that

$$\mathfrak{N}_g(L) \sim \frac{e^{hL}}{hL}$$

2010 *Mathematics Subject Classification.* Primary 20F65, Secondary 57M, 37B, 37D.

The first author was supported by the individual NSF grants DMS-1405146 and DMS-1710868. Both authors acknowledge support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: GEometric structures And Representation varieties" (the GEAR Network).

Every closed geodesic in $\mathcal{M}_g$ uniquely corresponds to the $MCG(\Sigma_g)$-conjugacy class of a pseudo-Anosov element φ of the mapping class group $MCG(\Sigma_g)$; the length of that closed geodesic is equal to the translation length of φ along the (unique) Teichmüller geodesic axis $A_\varphi \subseteq \mathcal{T}(\Sigma_g)$. That translation length, in turn, is equal to $\log \lambda(\varphi)$, where $\lambda(\varphi)$ is the *dilatation* or *stretch factor* of φ. Thus, $\mathfrak{N}_g(L)$ is equal to the number of $MCG(\Sigma_g)$-conjugacy classes of pseudo-Anosov elements $\varphi \in MCG(\Sigma_g)$ with $\log \lambda(\varphi) \leq L$.

It is natural to ask about the $\mathrm{Out}(F_r)$ situation. There the *Culler-Vogtmann Outer space* CV_r provides a counterpart to the Teichmüller space of a closed surface and the quotient $\mathcal{M}_r$ of CV_r under the $Out(F_r)$ action provides a counterpart to the moduli space $\mathcal{M}_g$. Instead of marked hyperbolic metrics on a surface, points of CV_r are "marked metric graph structures" for F_r. There is also a natural asymmetric geodesic metric d on CV_r that provides a substitute for the Teichmüller metric. See §2 below for additional references and details. In the $Out(F_r)$ setting, the main analog of the notion of being pseudo-Anosov is the notion of a "fully irreducible" element of $\mathrm{Out}(F_r)$. Namely, $\varphi \in \mathrm{Out}(F_r)$ is called *fully irreducible* if no positive power fixes the conjugacy class of a nontrivial proper free factor. Let X be a free basis of F_r. For a fully irreducible $\varphi \in \mathrm{Out}(F_r)$ and $1 \neq w \in F_r$ such that w does not represent a φ-periodic conjugacy class in F_r, the limit

$$\lambda(\varphi) := \lim_{n\to\infty} \sqrt[n]{||\varphi^n(w)||_X}$$

exists and is independent of w and X. This limit is called the *stretch factor* of φ. Again, see §2 below for more details. Every fully irreducible $\varphi \in \mathrm{Out}(F_r)$ admits an invariant geodesic axis $A_\varphi \subseteq CV_r$ on which φ acts with translation length $\log \lambda(\varphi)$. However, unlike in the Teichmüller space setting, such an axis is in general highly nonunique. The collection of all axes of φ is encoded by an important object called the *axis bundle* of φ (see [HM11]).

Counting the conjugacy classes of fully irreducibles $\varphi \in \mathrm{Out}(F_r)$ with a given bound on $\log \lambda(\varphi)$ is a considerably more difficult problem than the (already deep) corresponding problem for the $MCG(\Sigma_g)$. The main complication is that the CV_r does not admit any of the nice local analytic structures present in the Teichmüller space setting, precluding the use of classical methods of ergodic theory in analyzing the geodesic flow dynamics. Indeed, we will see below that the counting results we obtain in the $Out(F_r)$ setting exhibit new behavior, not present in classical hyperbolic dynamical systems. Since the local geometry of CV_r does not help, one may be inclined to try to use the several known $\mathrm{Out}(F_r)$-conjugacy class invariants. Unfortunately, they are not well suited for counting problems. The value of the stretch factor $\lambda(\varphi)$ is one such invariant of the conjugacy class of a fully irreducuble $\varphi \in \mathrm{Out}(F_r)$. It turns out that counting distinct values of $\lambda(\varphi)$ (without multiplicities) is ill-suited for the purposes of counting $\mathrm{Out}(F_r)$-conjugacy classes in the following sense. In Remark 5.7 we show that for $r \geq 2$ there is a constant $h = h(r) > 1$ such that

$$\#\{\lambda(\varphi) | \varphi \in \mathrm{Out}(F_r) \text{ is fully irreducible with } \log \lambda(\varphi) \leq L\} \leq h^L.$$

This fact stands in sharp contrast with the double exponential growth established in Theorem 1.1 below. Other $Out(F_r)$-conjugacy class invariants of fully irreducibles, such as the index, the index list, and the ideal Whitehead graph of φ (see [HM11]), only admit finitely many values for a given rank r and thus are also not suitable for counting purposes.

The first result about counting conjugacy classes of fully irreducibles was obtained in a recent paper of Hull and Kapovich [HK17]. They proved, roughly, that for $r \geq 3$ the number of distinct $\mathrm{Out}(F_r)$-conjugacy classes $[\varphi]$ of fully irreducibles φ coming from a ball of radius L in the Cayley graph of $\mathrm{Out}(F_r)$, and with $\log \lambda(\varphi)$ on the order of L, grows exponentially in L. The paper also provides an informal heuristic argument for why one might expect that the total number of $\mathrm{Out}(F_r)$-conjugacy classes $[\varphi]$ of fully irreducibles φ with $\log \lambda(\varphi) \leq L$ grows doubly exponentially in L. Here we prove that this is indeed the case. Our main result is:

Theorem 1.1. *For each integer $r \geq 3$, there exist constants $a = a(r) > 1, b = b(r) > 1, c = c(r) > 1$ so that: For $L \geq 1$, let $\mathfrak{N}_r(L)$ denote the number of $Out(F_r)$-conjugacy classes of fully irreducibles $\varphi \in Out(F_r)$ with $\log \lambda(\varphi) \leq L$. Then there exists an $L_0 \geq 1$ such that for all $L \geq L_0$ we have*

$$c^{e^L} \leq \mathfrak{N}_r(L) \leq a^{b^L}.$$

Therefore, c^{e^L} bounds below the number of closed geodesics in $\mathcal{M}_r$ of length bounded above by L.

Here, as in the results above, e is the base of the natural logarithm. By a closed geodesic in $\mathcal{M}_r$ we mean the image in $\mathcal{M}_r$ of a periodic geodesic in CV_r. Note that not all closed geodesics in $\mathcal{M}_r$ come from axes of fully irreducibles, since there exist nonirreducible elements in $\mathrm{Out}(F_r)$ that admit periodic geodesic lines in CV_r. However, it is known, by a combination of results of Besvina and Feighn [BF14] and of Dowdall and Taylor [DT15] that a φ-periodic geodesic $A_\varphi \subseteq CV_r$ is "contracting" with respect to the asymmetric Lipschitz metric d on CV_r if and only if $\varphi \in CV_r$ is fully irreducible. Therefore, Theorem 1.1 can be interpreted as providing double exponential lower and upper bounds on the number of equivalence classes of "contracting" closed geodesics of length $\leq L$ in $\mathcal{M}_r$, where loops from axes of the same outer automorphism are deemed equivalent.

The case of rank $r = 2$ is special, and Theorem 1.1 does not apply. The group $Out(F_2)$ is commensurable with the mapping class group $MCG(\Sigma_{1,1})$ of the punctured torus $\Sigma_{1,1}$ and with the group $SL(2, \mathbb{Z})$. The Teichmüller space $\mathcal{T}(\Sigma_{1,1})$ is the hyperbolic plane $\mathbb{H}^2$, with a faithful discrete isometric action of $MCG(\Sigma_{1,1})$ as a nonuniform lattice. Counting $\mathfrak{N}_2(L)$ amounts to (up to correcting for the commensurability effects) computing the number of conjugacy classes in that lattice of translation length $\leq L$. By the classic counting results, this produces exponential asymptotics for $\mathfrak{N}_2(L)$, rather than the double exponential asymptotics displayed in Theorem 1.1.

We briefly discuss the idea of the proof of Theorem 1.1 here. Every fully irreducible $\varphi \in \mathrm{Out}(F_r)$ is the induced map of fundamental groups for an "efficient" graph map $f : \Gamma \to \Gamma$, called a *train track map* [BH92]. Here Γ is *marked* by an identification $\pi_1(\Gamma) \cong F_r$. The stretch factor $\lambda(\varphi)$ is then equal to the Perron-Frobenius eigenvalue $\lambda(f)$ of the transition matrix of f. We introduce a new $\mathrm{Out}(F_r)$-conjugacy class invariant $\mathbf{U}(\varphi)$ which counts the number of distinct combinatorial types of unmarked train track representatives $f : \Gamma \to \Gamma$ of a fully irreducible $\varphi \in \mathrm{Out}(F_r)$ such that the underlying graph Γ is an r-rose R_r, i.e. graph with a single vertex and betti number $b_1(\Gamma) = r$. A priori, the upper bound for the cardinality of $\mathbf{U}(\varphi)$ is double exponential in $\log \lambda(\varphi)$, see Lemma 5.6 below. However, there is an important class of fully irreducibles for which this bound is much better. These are the so-called *lone axis* fully irreducibles, which are so called "ageometric" fully irreducibles $\varphi \in \mathrm{Out}(F_r)$ with a unique invariant axis A_φ in CV_r. Lone axis fully irreducibles were introduced and studied by Mosher and Pfaff in [MP16]. Mosher and Pfaff provided an "ideal Whitehead graph" $\mathcal{IW}(\varphi)$ criterion in [MP16] for an ageometric fully irreducible φ to have a lone axis. Here we show that, if φ is a lone axis fully irreducible with a train track representative $f : R_r \to R_r$, then $\#\mathbf{U}(\varphi) \leq ||f||$, where $||f||$ is the sum of the lengths of the edge-paths $f(e)$ as e varies over the edges of R_r.

Now let $r \geq 3$ and let $X = \{x_1, \dots x_r\}$ be a free basis of F_r. For a "random" positive word $w(x_2, \dots, x_r)$ of length e^L we construct an explicit positive automorphism ψ_w of F_r such that, when viewed as a train track map g_w on the rose R_r, it satisfies $||g_w|| \approx e^L$. We then precompose g_w with another positive train track map $\eta : R_r \to R_r$ to get a train track map $f_w = g_w \circ \eta : R_r \to R_r$ representing an outer automorphism $\varphi_w \in \mathrm{Out}(F_r)$. Denote the set of all such φ_w by $\mathcal{S}_r$. The fact that η does not depend on w and L means that $||f_w|| \leq Ce^L$. The number of distinct "random" positive words $w(x_2, \dots, x_r)$ of length e^L is on the order of $(r-1)^{e^L}$, which gives us on the order of $(r-1)^{e^L}$ combinatorially distinct unmarked train track maps $f_w : R_r \to R_r$. The key step is to choose η in such a way that for all w as above, φ_w is a lone axis ageometric fully irreducible. Technically, this is the hardest part of the proof since satisfying the lone axis property for φ_w

requires, among other things, that f_w have no periodic Nielsen paths (PNPs). Here we rely on train track automata ($\mathcal{ID}$ diagrams) and PNP prevention technology developed by Pfaff in [Pfa12, Pfa13]. Once we know that each φ_w is lone axis, the above estimate for the size of $\mathbf{U}$ implies that $\#U(\varphi_w) \leq Ce^L$. Thus the maps f_w give approximately $(r-1)^{e^L}$ combinatorially distinct train track maps on roses representing ageometric fully irreducibles φ_w, each with $\#U(\varphi_w) \leq Ce^L$. Therefore,

$$(\ddagger) \qquad \#\mathcal{S}_r \geq const \frac{(r-1)^{e^L}}{Ce^L} \geq_{L\to\infty} (r-1.5)^{e^L}.$$

The fact that each φ_w has a train track representative f_w with $||f_w|| \leq Ce^L$ implies that $\log \lambda(\varphi_w) \leq L + \log C$, which, together with ($\ddagger$), leads to the lower bound in Theorem 1.1. The upper bound in Theorem 1.1 is much easier, and is obtained by a Perron-Frobenius counting argument estimating from above the number of train track maps f with $\log \lambda(f) \leq L$; see Lemma 5.6 below.

Acknowledgements. We are grateful to Yael Algom-Kfir, Michael Hull, and Paul Schupp for useful discussions. Both authors are also grateful to the city Santa Barbara for its inspiring beauty, and to the firefighters who saved Santa Barbara from the destructive flames of the Thomas fire.

2. Definitions and Background

We assume throughout this paper that $r \geq 3$ is an integer, and that F_r is the rank-r free group with a fixed generating set $X = \{x_1, \dots, x_r\}$. R_r will denote the r-petaled rose, i.e. the graph with r edges and a single vertex. We fix an orientation on R_r and an identification of each positive edge e_i of R_r with an element x_i of X.

2.1. **Train track maps & (fully) irreducible outer automorphisms.**

This paper follows the conventions and formalism regarding graphs and graph maps explained in detail in [DKL15]. In particular, unless specified otherwise, graphs are 1-dimensional CW-complexes equipped with a "linear atlas of charts" on edges, and all graph maps, topological representatives, and train track maps are assumed to be "linear graph maps", in the terminology of [DKL15]. Basically these assumptions translate to working in the PL category, ruling out various pathologies for fixed and periodic points of train track maps in relation to Nielsen paths and periodic Nielsen paths. We refer the reader to [DKL15] for more details (not important for us here).

Unless otherwise indicated, Γ and Γ' are graphs with no vertices of valence 1 or 2. $E\Gamma$ will denote the edge set of Γ and $V\Gamma$ will denote the vertex set.

Definition 2.1 (Graph maps & train track maps)**.** We call a continuous map of graphs $g\colon \Gamma \to \Gamma'$ a *graph map* if it takes vertices to vertices and is locally injective on the interior of each edge. A graph map $g\colon \Gamma \to \Gamma$ is a *train track map* if g is a homotopy equivalence and if for each $k \geq 1$ the map g^k is locally injective on edge interiors. We call the train track map g *expanding* if for each edge $e \in E\Gamma$ we have that $|g^n(e)| \to \infty$ as $n \to \infty$, where for a path γ we use $|\gamma|$ to denote the number of edges γ traverses (with multiplicity). We call g *irreducible* if it has no proper invariant subgraph with a noncontractible component.

If $\varphi \in Out(F_r)$, and Γ is equipped with a *marking* (i.e. a homotopy equivalence $m\colon R_r \to \Gamma$), and $g\colon \Gamma \to \Gamma$ is a graph map such that $g_* = \varphi$, then we say *g represents φ*.

Note that, via our identification of $E\Gamma$ with the free basis X, from an automorphism $\Phi \in Aut(F_r)$, we obtain an *induced* graph map sending e_i to $e_{i,1} \dots e_{i,k}$ where $\Phi(x_i) = x_{i,1} \dots x_{i,k}$. (This will be a representative of φ, the outer class of Φ.) We may sometimes blur the distinction between an automorphism and the graph map it induces.

Definition 2.2 (Directions)**.** Let $x \in \Gamma$. The *directions* at x are the germs of initial segments of edges emanating from x. For each directed edge $e \in E\Gamma$, we let $D(e)$, or just e, denote the initial direction of e. For an edge-path $\gamma = e_1 \dots e_k$, define $D\gamma := D(e_1)$. Let $g\colon \Gamma \to \Gamma$ be a graph map. Then denote by Dg the map of directions induced by g, i.e. $Dg(d) = D(g(e))$ for $d = D(e)$. A direction d is *periodic* if $Dg^k(d) = d$ for some $k > 0$ and *fixed* when $k = 1$.

Definition 2.3 (Turns)**.** An unordered pair of directions $\{d_i, d_j\}$ at a common vertex is called a *turn*, and a *degenerate turn* if $d_i = d_j$. An edge-path containing no degenerate turns is called *tight*. For a path $\gamma = e_1 e_2 \dots e_{k-1} e_k$ in Γ where e_1 and e_k may be partial edges, we say γ *takes* $\{\overline{e_i}, e_{i+1}\}$ for each $1 \leq i < k$. For both edges and paths we more generally use an "overline" to denote a reversal of orientation. A path γ is *g-legal* if each turn of γ is g-legal.

Let $g\colon \Gamma \to \Gamma$ be a graph map. Denote also by Dg the map induced by Dg on the turns of Γ. A turn τ is called g-*prenull* if $Dg(\tau)$ is degenerate. The turn τ is called an *illegal turn* for g if $Dg^k(\tau)$ is degenerate for some k and a *legal turn* otherwise. A turn T in Γ is *g-taken* if there exists an edge e so that $g(e)$ takes T. We use $\mathcal{T}(g)$ to denote the set of g-taken turns and define $\mathcal{T}_\infty := \cup_{k \geq 1} \mathcal{T}(g^k)$.

Definition 2.4 (Transition matrix $M(g)$, Perron-Frobenius matrix, Perron-Frobenius eigenvalue)**.** The *transition matrix* $M(g)$ of a train track map $g\colon \Gamma \to \Gamma$ is the square $|E\Gamma| \times |E\Gamma|$ matrix (a_{ij}) such that a_{ij}, for each i and j, is the number of times $g(e_i)$ traverses e_j in either direction. A transition matrix $M = [a_{ij}]$ is *Perron-Frobenius (PF)* if there exists an N such that M^k is strictly positive, for all $k \geq N$. By Perron-Frobenius theory, we know that each such matrix has a unique eigenvalue of maximal modulus and that this eigenvalue is real and > 1. This eigenvalue is called the *Perron-Frobenius (PF) eigenvalue* of M and for $M(g)$ is denoted $\lambda(g)$.

Definition 2.5 (Stretch factor of a fully irreducible)**.** Given a free basis X of F_r and element $w \in F_r$, denote by $||w||_X$ the cyclically reduced length of w with respect to X. For a fully irreducible $\varphi \in \mathrm{Out}(F_r)$, free basis X, and $1 \neq w \in F_r$ such that the conjugacy class $[w]$ is not φ-periodic, it is known [Bog08] that the limit

$$\lim_{n\to\infty} \sqrt[n]{||\varphi^n(w)||_X}$$

exists and is independent of X and w. This limit is called the *stretch factor* of φ and is denoted $\lambda(\varphi)$. If $g : \Gamma \to \Gamma$ is a train track representative of φ then $\lambda(g) = \lambda(\varphi)$ (see, for example, [Bog08]).

2.2. **Periodic Nielsen paths.**

Definition 2.6 (Nielsen paths & rotationless powers)**.** Let $g\colon \Gamma \to \Gamma$ be an expanding irreducible train track map. Bestvina and Handel [BH92] define a nontrivial immersed path ρ in Γ to be a *periodic Nielsen path (PNP)* if $g^R(\rho) \cong \rho$ rel endpoints for some power $R \geq 1$ (and just a *Nielsen path (NP)* if $R = 1$). A NP ρ is called *indivisible* (hence is an "iNP") if it cannot be written as $\rho = \gamma_1\gamma_2$, where γ_1 and γ_2 are themselves NPs.

By [FH11, Corollary 4.43], for each $r \geq 2$, there exists an $R(r) \in \mathbb{N}$ such that for each expanding irreducible train track representative g of each outer automorphism $\varphi \in Out(F_r)$, each PNP for g is an NP for $g^{R(r)}$. This power R is called the *rotationless* power.

We remark that iNPs have a specific structure, described in [BH92, Lemma 3.4]:

Proposition 2.7. *Let $g\colon \Gamma \to \Gamma$ be an expanding irreducible train track map. Then every iNP ρ in Γ has the form $\rho = \overline{\rho_1}\rho_2$, where ρ_1 and ρ_2 are nondegenerate legal paths sharing their initial vertex $v \in \Gamma$ and such that the turn at v between ρ_1 and ρ_2 is an illegal nondegenerate turn for g.*

Definition 2.8 (Ageometric)**.** A fully irreducible outer automorphism is called *ageometric* if it has a train track representative with no PNPs.

2.3. **Whitehead graphs.** Throughout this subsection, $g\colon R_r \to R_r$ will be a PNP-free expanding irreducible train track map and v will be the vertex of R_r. More general definitions can be found in [HM11] or [MP16], with explanations in [Pfa12] of the reduction to our setting.

Definition 2.9 (Whitehead graphs & index)**.** The *local Whitehead graph* $LW(g)$ has a vertex for each direction at v and an edge connecting the vertices corresponding to a pair of directions $\{d_1, d_2\}$ when $\{d_1, d_2\} \in \mathcal{T}_\infty$. The *stable Whitehead graph* $SW(g)$ is the subgraph of $LW(v;\Gamma)$ obtained by restricting to the vertices of $LW(g)$ corresponding to periodic directions.

If g further represents a fully irreducible $\varphi \in Out(F_r)$, then the *ideal Whitehead graph* $\mathcal{IW}(\varphi)$ *of* φ is isomorphic to $SW(g)$. Justification of this being an outer automorphism invariant can be found in [HM11, Pfa12]. From the ideal Whitehead graph, one can obtain the *rotationless index* $i(\varphi) := 1 - \frac{k}{2}$, where k is the number of vertices of $\mathcal{IW}(\varphi)$.

We will use the following lemma whose proof, while not explicitly given in either [KP15, Lemma 3.7] or [Pfa12], is of the flavor of proofs in each.

Lemma 2.10. *Suppose that $h_1, \cdots, h_n$ are train track maps so that, for each i, if $h_i(e) = e_1 \dots e_\ell$, then $h_{i+1}(e_1) \dots h_{i+1}(e_\ell)$ is tight (indices here are viewed in $\mathbb{Z} / n\mathbb{Z}$). Suppose further that $h_1 \circ \cdots \circ h_n$ is a train track map. Then*

$$\mathcal{T}(h_1 \circ \cdots \circ h_n) = [\mathcal{T}(h_1)] \bigcup_{k=2}^{n} [D(h_1 \circ \cdots \circ h_{k-1})(\mathcal{T}(h_k))].$$

2.4. **Full irreducibility criterion.** Proposition 2.11 is stated as such in [AKKP17]. It is essentially [Pfa13, Proposition 4.1], with the added observation that a fully irreducible outer automorphism with a PNP-free train track representative is in fact ageometric (by definition). Kapovich [Kap14] has a related result.

Proposition 2.11 ([AKKP17])**.** *(*The Ageometric Full Irreducibility Criterion (FIC)*) Let $g\colon \Gamma \to \Gamma$ be a PNP-free, irreducible train track representative of $\varphi \in Out(F_r)$. Suppose that $M(g)$ is Perron-Frobenius and that all the local Whitehead graphs are connected. Then φ is an ageometric fully irreducible outer automorphism.*

3. Fold lines in outer space

For each integer $r \geq 3$, we use CV_r to denote the rank-r Culler-Vogtmann outer space (as defined in [CV86]). See [BSV14] for a nice survey.

Definition 3.1 (Stallings folds)**.** [Sta83]. Let $g\colon \Gamma \to \Gamma'$ be a homotopy equivalence graph map. Let $e_1' \subset e_1$ and $e_2' \subset e_2$ be maximal, initial, nontrivial subsegments of edges e_1 and e_2 emanating from a common vertex and satisfying that $g(e_1') = g(e_2')$ as edge paths and that the terminal endpoints of e_1' and e_2' are distinct points in $g^{-1}(V\Gamma)$. Redefining Γ to have vertices at the endpoints of e_1' and e_2' if necessary, one can obtain a graph Γ_1 by identifying the points of e_1' and e_2' that have the same image under g, a process called *folding.*

Definition 3.2 (Stallings fold decomposition)**.** Stallings [Sta83] also showed that if $g\colon \Gamma \to \Gamma'$ is a homotopy equivalence graph map, then g factors as a composition of folds and a final homeomorphism. We call such a decomposition a *Stallings fold decomposition.* It can be obtained as follows: At an prenull turn for $g\colon \Gamma \to \Gamma'$, one can fold two maximal initial segments having the same image in Γ' to obtain a map $\mathfrak{g}_1\colon \Gamma_1 \to \Gamma'$ of the quotient graph Γ_1. The process can be repeated for $\mathfrak{g}_1$

and recursively. If some $\mathfrak{g}_k\colon \Gamma_{k-1} \to \Gamma$ has no prenull turn, then $\mathfrak{g}_k$ will be a homeomorphism and the fold sequence is complete. The process is known to terminate.

$$\begin{tikzcd}
\Gamma_0 \arrow[r, "g_1"'] \arrow[rrrr, bend left, "g=\mathfrak{g}_0"] & \Gamma_1 \arrow[r, "g_2"'] \arrow[rrr, bend left, "\mathfrak{g}_1"] & \Gamma_2 \arrow[r, "g_3"'] \arrow[rr, bend left, "\mathfrak{g}_2"] & \dots \arrow[r, "g_n"'] & \Gamma_n = \Gamma'
\end{tikzcd}$$

Figure 1. Constructing a Stallings folds decomposition

Remark 3.3. *Another way to view the process, important for the proof of Lemma 5.4, is to "label" each edge e in Γ by its image path $f(e)$ and to iteratively perform folds of identically labeled edge segments emanating from a common vertex.*

In [Sko89], Skora interpreted a Stallings fold decomposition for a graph map homotopy equivalence $g\colon \Gamma \to \Gamma'$ as a sequence of folds performed continuously. By repeating this procedure, one obtains a geodesic in CV_r called a *periodic fold line* for g. It is proved in [AKKP17, Lemma 2.27] that, if g is a train track map, then the periodic fold line is in fact a geodesic

Remark 3.4. *In general, each fully irreducible $\varphi \in CV_r$ has many train track representatives, each of which can have several distinct Stallings fold decompositions (and hence several associated periodic fold lines).*

3.1. Lone axis fully irreducible outer automorphism.

In [HM11], Handel and Mosher define the axis bundle for an ageometric fully irreducible outer automorphism φ. [HM11] contains three equivalent definitions. Definition 3.5 below provides a fourth equivalent definition (proving the equivalence of this definition to the others is straight-forward).

Definition 3.5 (Axis bundle $\mathcal{A}_\varphi$)**.** Let $\varphi \in Out(F_r)$ be an ageometric fully irreducible outer automorphism. Then $\mathcal{A}_\varphi$ is the closure of the union of the images of the periodic fold lines for train track representatives of positive powers of φ.

The following theorem is proved in [MP16]. The statement given here is in fact a combination of [MP16, Theorem 4.6], [MP16, Theorem 4.7], and [MP16, Lemma 4.5].

Theorem 3.6 ([MP16])**.** *Let $\varphi \in Out(F_r)$ be an ageometric fully irreducible outer automorphism. Then φ is a lone axis fully irreducible outer automorphism if and only if*

(1) the rotationless index satisfies $i(\varphi) = \frac{3}{2} - r$ and
(2) no component of the ideal Whitehead graph $\mathcal{IW}(\varphi)$ has a cut vertex.

In this case it is a unique periodic fold line. In particular, it is the unique periodic fold line for each train track representative of each power of φ.

4. The Automorphisms

We continue assuming that $r \geq 3$, that $F_r = F(x_1, \dots, x_r)$, and that R_r is the r-petaled rose. We may blur the distinction between an automorphism and its induced map on the rose.

Definition 4.1 (Full word)**.** We say that a positive word $w(x_2, \dots, x_r)$ is *full* if for all $2 \leq i, j \leq r$ the word $x_i x_j$ occurs as a subword of w. That is, a positive word $w(x_2, \dots, x_r)$ is full if and only if it contains each turn $\{d_1, d_2\}$ with $d_1 \in \{x_2, \dots, x_r\}$ and $d_2 \in \{\overline{x_2}, \dots, \overline{x_r}\}$.

Definition 4.2 (g_w)**.** Let $r \geq 3$ and let $w(x_2, \dots, x_r)$ be a full positive word in $x_2, \dots, x_r$ starting with x_{r-1} and ending with x_2. We then define graph maps $g_w : R_r \to R_r$ by:

$$g_w(x_k) = \begin{cases} x_{k+1} \quad if \quad 1 \leq k \leq r-1 \\ x_1 w(x_2, \dots, x_r) \quad if \quad k = r \end{cases}$$

Definition 4.3 (g_k, $g_{k,i}$)**.** For each $\varepsilon \in \{0, 1\}$, we define the following automorphisms (all generators whose images are not explicitly given are fixed):

$$g_{1+\varepsilon} : x_1 \mapsto x_1 x_r, \quad g_{2+\varepsilon} : x_r \mapsto x_r x_1, \quad g_{3+\varepsilon} : x_r \mapsto x_r x_{r-1}$$

$$g_{4+\varepsilon} : x_{r-1} \mapsto x_{r-1} x_r, \quad g_{5+\varepsilon} : x_{r-1} \mapsto x_{r-1} x_1, \quad g_{6+\varepsilon} : x_1 \mapsto x_1 x_{r-1}.$$

Denoting the identity map on R_r by id_{R_r}, for each $1 \leq i, k \leq 12$, let

$$g_{k,i} = \begin{cases} g_k \circ \dots \circ g_i \text{ if } k > i \\ g_k \text{ if } i = k \\ id_{R_r} \text{ if } i > k \end{cases}$$

Remark 4.4. *The sequence of g_i was constructed using the $\mathcal{ID}$ diagrams of* [Pfa12].

Remark 4.5 (Compositions are train track maps)**.** *Notice that, since each g_w and each g_i represent positive automorphisms of F_r, any composition of them is also a positive automorphism. Hence, any such composition induces a train track map on the rose R_r.*

Lemma 4.6. *Suppose that $w(x_2, \dots, x_r)$ is a full positive word in $x_2, \dots, x_r$ starting with x_{r-1} and ending with x_2. Let $g := g_w \circ g_{12,1} : R_r \to R_r$. Then $LW(g)$ consists of the edge connecting the pair $\{\overline{x_1}, x_{r-1}\}$ together with the complete bipartite graph on the partition $\{\{x_1, \dots, x_r\}, \{\overline{x_2}, \dots, \overline{x_r}\}\}$.*

Proof. We begin with the following observations:

(1)
$$\mathcal{T}_\infty(g) = \bigcup_{\ell \geq 1} [D(g^{\ell-1} \circ g_w)(\mathcal{T}(g_{12,1})) \cup Dg^{\ell-1}(\mathcal{T}(g_w))]$$

(2) Dg_w is defined by $x_k \mapsto x_{(k+1 \text{ mod r})}$ for $1 \leq k \leq r$, and $\overline{x_k} \mapsto \overline{x_{k+1}}$ for $1 \leq k \leq r-1$, and $\overline{x_r} \mapsto \overline{x_2}$.

(3) $D(g_{12} \circ \dots \circ g_1)$ is the identity map apart from $\overline{x_1} \mapsto \overline{x_r}$

(4) Dg is defined by $x_k \mapsto x_{(k+1 \text{ mod r})}$ for $1 \leq k \leq r$, and $\overline{x_k} \mapsto \overline{x_{k+1}}$ for $1 \leq k \leq r-1$ and $\overline{x_r} \mapsto \overline{x_2}$.

(5) $\mathcal{T}(g_w) = \{\{d_1, d_2\} \mid d_1 \in \{x_2, \dots, x_r\} \;\&\; d_2 \in \{\overline{x_2}, \dots, \overline{x_r}\}\} \cup \{\{\overline{x_1}, x_{r-1}\}\}$

(6) $\mathcal{T}(g_{12,1}) = \{\{d_1, d_2\} \mid d_1 \in \{x_1, x_{r-1}, x_r\} \;\&\; d_2 \in \{\overline{x_{r-1}}, \overline{x_r}\}\} \cup \{\{\overline{x_1}, x_{r-1}\}\}$ (see, for example, [Pfa15])

Since $Dg(\{\overline{x_{k-1}}, x_r\}) = \{\overline{x_k}, x_1\}$ for all $2 \leq k \leq r$ and $\{\overline{x_1}, x_{r-1}\} \in \mathcal{T}(g_w)$, we have

$$\{\{x_1, d\} \mid d \in \{\overline{x_2}, \dots, \overline{x_r}\}\} \cup \{\{\overline{x_1}, x_{r-1}\}\} \subset \bigcup_{\ell \geq 1} [Dg^{\ell-1}(\mathcal{T}(g_w))] \subset \mathcal{T}_\infty.$$

Together with $\mathcal{T}(g_w) = \{\{d_1, d_2\} \mid d_1 \in \{x_2, \dots, x_r\} \ \&\ d_2 \in \{\overline{x_2}, \dots, \overline{x_r}\}\} \cup \{\{\overline{x_1}, x_{r-1}\}\} \subset \mathcal{T}_\infty$, this says

$$\{\{d_1, d_2\} \mid d_1 \in \{x_1, \dots, x_r\} \ \&\ d_2 \in \{\overline{x_2}, \dots, \overline{x_r}\}\} \cup \{\{\overline{x_1}, x_{r-1}\}\} \subset \mathcal{T}_\infty.$$

Since $\overline{x_1}$ is not in the image of Dg or Dg_w, we then have

$$\mathcal{T}_\infty = \{\{d_1, d_2\} \mid d_1 \in \{x_1, \dots, x_r\} \ \&\ d_2 \in \{\overline{x_2}, \dots, \overline{x_r}\}\} \cup \{\{\overline{x_1}, x_{r-1}\}\}.$$

□

In the proof of Lemma 4.7, the procedure for showing that no iNPs exist is similar to that in [Pfa13], [HM11, Example 3.4], or [Pfa15].

Lemma 4.7. *Suppose that $w(x_2, \dots, x_r)$ is a full positive word starting with x_{r-1} and ending with x_2. Then $g_w \circ g_{12,1}$ represents an ageometric fully irreducible outer automorphism of F_r.*

Proof. By Remark 4.5, g is a train track map. It is expanding since the image of each edge under $g_{12,1}$ contains every edge (including itself), the g_w-image of each of these edges contains at least one edge, and the g_w-image of each edge is tight. The transition matrix $M(g)$ is PF since the image of each edge under $g_{12,1}$ contains every edge (including itself) and g_w is surjective.

To show that g represents an ageometric fully irreducible outer automorphism, we prove that g additionally satisfies the remaining conditions of Proposition 2.11, i.e $LW(g)$ is connected and g is PNP-free. Since $LW(g)$ is connected by Lemma 4.6, we now show that g has no PNPs.

Suppose for the sake of contradiction that g has a PNP. Taking the rotationless power g^R, this gives an NP. Let $\rho = \overline{\rho_1}\rho_2$ be an iNP for g^R in the decomposition of ρ into iNPs, where $\rho_1 = e_1 \dots e_m$ and $\rho_2 = e'_1 \dots e'_{m'}$ are edge paths (with possibly e_m and $e'_{m'}$ being partial edges) and with illegal turn $T_i = \{D(e_1), D(e'_1)\} = \{d_1, d'_1\}$. Notice first that each turn of ρ_1 and of ρ_2 must be in $\mathcal{T}_\infty(g^R)$. And that $\{\overline{x_1}, x_r\} \notin \mathcal{T}_\infty(g)$. Also notice that, for ρ to be an iNP, for each $1 \le i \le 12$, we have that $(g_{i,1})_\#(\rho)$ must be prenull with respect to $g^{R-1} \circ g_w \circ g_{12} \circ \dots \circ g_{i+1}$.

Since $\{\overline{x_1}, \overline{x_r}\}$ is the only illegal turn for g^R, it would be necessary that (without loss of generality) $e_1 = \overline{x_1}$ and $e'_1 = \overline{x_r}$. Now:

$$g_1(\rho_1) = \bar{x}_r \bar{x}_1 g_1(e_2) \dots g_1(e_m)$$
$$g_1(\rho_2) = \overline{x_r} g_1(e'_2) \dots g_1(e'_m).$$

For ρ to be an iNP, we need that $\{Dg_1(e'_2), \overline{x_1}\}$ is either degenerate or the illegal turn $\{\overline{x_1}, \overline{x_r}\}$ for g_2. Since $\overline{x_1}$ is not in the image of Dg_1, this leaves that $Dg_1(e'_2) = \overline{x_r}$. This would happen precisely when either $e'_2 = \overline{x_1}$ or $e'_2 = \overline{x_r}$. However, if $e'_2 = \overline{x_1}$, then ρ_2 would contain turns not in $\mathcal{T}_\infty(g)$. Hence, $e'_2 = \overline{x_r}$. Now:

$$g_{2,1}(\rho_1) = \bar{x}_1 \bar{x}_r \bar{x}_1 g_{2,1}(e_2) \dots g_{2,1}(e_m)$$
$$g_{2,1}(\rho_2) = \bar{x}_1 \bar{x}_r \bar{x}_1 \bar{x}_r g_{2,1}(e'_3) \dots g_{2,1}(e'_m).$$

So we need that $\{Dg_{2,1}(e_2), \overline{x_r}\}$ is either degenerate or the illegal turn $\{\overline{x_r}, \overline{x_{r-1}}\}$ for g_3. Since $\overline{x_r}$ is not in the image of $Dg_{2,1}$, this leaves that $Dg_{2,1}(e_2) = \overline{x_{r-1}}$, i.e. $e_2 = \overline{x_{r-1}}$. Now:

$$g_{3,1}(\rho_1) = \bar{x}_1 \overline{x_{r-1}} \bar{x}_r \bar{x}_1 \overline{x_{r-1}} g_{3,1}(e_3) \dots g_{3,1}(e_m)$$
$$g_{3,1}(\rho_2) = \bar{x}_1 \overline{x_{r-1}} \bar{x}_r \bar{x}_1 \overline{x_{r-1}} \bar{x}_r g_{3,1}(e'_3) \dots g_{3,1}(e'_m).$$

So we need that $\{Dg_{3,1}(e_3), \overline{x_r}\}$ is either degenerate or the illegal turn $\{\overline{x_r}, \overline{x_{r-1}}\}$ for g_4. Since $\overline{x_r}$ is not in the image of $Dg_{3,1}$, this leaves that $Dg_{3,1}(e_3) = \overline{x_{r-1}}$, i.e. $e_3 = \overline{x_{r-1}}$. Now:

$$g_{4,1}(\rho_1) = \bar{x}_1 \bar{x}_r \overline{x_{r-1}} \bar{x}_r \bar{x}_1 \bar{x}_r \overline{x_{r-1}} \bar{x}_r \overline{x_{r-1}} g_{4,1}(e_4) \dots g_{4,1}(e_m)$$
$$g_{4,1}(\rho_2) = \bar{x}_1 \bar{x}_r \overline{x_{r-1}} \bar{x}_r \bar{x}_1 \bar{x}_r \overline{x_{r-1}} \bar{x}_r g_{4,1}(e'_3) \dots g_{4,1}(e'_m).$$

So we need that $\{Dg_{4,1}(e_3'), \overline{x_{r-1}}\}$ is either degenerate or the illegal turn $\{\overline{x_1}, \overline{x_{r-1}}\}$ for g_5. Since $\overline{x_{r-1}}$ is not in the image of $Dg_{4,1}$, this leaves that $Dg_{4,1}(e_3') = \overline{x_1}$, i.e. either $e_3' = \overline{x_r}$ or $e_3' = \overline{x_1}$. However, if $e_3' = \overline{x_1}$, then ρ_2 would contain turns not in $\mathcal{T}_\infty(g)$. Hence, $e_3' = \overline{x_r}$. Now:

$$g_{5,1}(\rho_1) = \bar{x_1}\bar{x_r}\bar{x_1}\overline{x_{r-1}}\bar{x_r}\bar{x_1}\bar{x_r}\bar{x_1}\overline{x_{r-1}}\bar{x_r}\bar{x_1}\overline{x_{r-1}}g_{5,1}(e_4)\dots g_{5,1}(e_m)$$

$$g_{5,1}(\rho_2) = \bar{x_1}\bar{x_r}\bar{x_1}\overline{x_{r-1}}\bar{x_r}\bar{x_1}\bar{x_r}\bar{x_1}\overline{x_{r-1}}\bar{x_r}\bar{x_1}\bar{x_r}\bar{x_1}\overline{x_{r-1}}\bar{x_r}g_{5,1}(e_4')\dots g_{5,1}(e_m').$$

This tells us that the illegal turn for g_7 would have to be $\{\overline{x_{r-1}}, \overline{x_r}\}$, but the illegal turn for g_7 is $\{\overline{x_1}, \overline{x_r}\}$. Thus, we have reached a contradiction and g can have no PNPs. □

Lemma 4.8. *Suppose that $w(x_2, \dots, x_r)$ is a full positive word in $x_2, \dots, x_r$ starting with x_{r-1} and ending with x_2. Let $g := g_w \circ g_{12,1}$ represent $\varphi_w \in Out(F_r)$. Then, $\mathcal{IW}(\varphi_w)$ is the complete bipartite graph on the partition $\{\{x_1, \dots, x_r\}, \{\overline{x_2}, \dots, \overline{x_r}\}\}$.*

Proof. Since there are no PNPs, we have $\mathcal{IW}(\varphi_w) \cong SW(g)$. Since all directions apart from $\overline{x_1}$ are periodic, $SW(g)$ is the graph obtain from $LW(g)$ by removing the vertex for $\overline{x_1}$. The result then follows from Lemma 4.6. □

Proposition 4.9. *Suppose that $w(x_2, \dots, x_r)$ is a full positive word, starting with x_{r-1} and ending with x_2. Then $g = g_w \circ g_{12,1}$ represents a lone axis ageometric fully irreducible $\varphi_w \in Out(F_r)$.*

Proof. By Lemma 4.7, φ is an ageometric fully irreducible outer automorphism. We can thus use Theorem 3.6 to prove that φ has a lone axis.

By Lemma 4.8, the ideal Whitehead graph has a single component, which has $2r - 1$ vertices. Thus, $i(\varphi_w) = \frac{3}{2} - r$, and so Theorem 3.6(1) is satisfied. Notice also that Lemma 4.8 implies that $\mathcal{IW}(\varphi_w)$ is a complete bipartite graph which has at least 2 vertices in each set of the partition. Hence, the only component of $\mathcal{IW}(\varphi_w)$ has no cut vertices. This implies that Theorem 3.6(2) is also satisfied. So φ_w has a lone axis, as desired. □

5. Counting

For the remainder of this paper $\log x$ will denote the natural logarithm of x.

Definition 5.1 ($\mathbf{U}(\varphi)$)**.** For $r \geq 3$ and $\varphi \in Out(F_r)$ be fully irreducible, $\mathbf{U}(\varphi)$ will denote the set of all unmarked train track representatives $f : R_r \to R_r$ of φ, considered as combinatorial graph maps. More precisely, $\mathbf{U}(\varphi)$ is the set of equivalence classes of train track representatives of φ based on an r-rose, equivalent when they differ by a change in marking and possibly a graph homeomorphism. That is, if $(f : \Gamma \to \Gamma, \alpha)$ and $(f' : \Gamma' \to \Gamma', \alpha')$ are train track representatives of φ on r-roses Γ and Γ', with markings α and α', these representatives are considered equivalent if there exists a homeomorphism $q : \Gamma' \to \Gamma$ such that $f' = q^{-1} \circ f \circ q$. (The existence of q means that f and f' are the same as combinatorial graph maps.)

Remark 5.2. *Note that the set $\mathbf{U}(\varphi)$ is possibly empty (since not every fully irreducible in $Out(F_r)$ has a train track representative on R_r). Moreover, $\mathbf{U}(\varphi) = \mathbf{U}(\psi^{-1}\varphi\psi)$ for each $\psi \in Out(F_r)$. Lemma 5.6 will imply that $\mathbf{U}(\varphi)$ is finite for every fully irreducible $\varphi \in Out(F_r)$.*

Remark 5.3. *Observe that if $\alpha, \beta \in \mathrm{Out}(F_r)$ are both fully irreducibles and representable by train track maps on roses, then α is conjugate to β in $\mathrm{Out}(F_r)$ if and only if $\mathbf{U}(\alpha) = \mathbf{U}(\beta)$, if and only if $\mathbf{U}(\alpha) \cap \mathbf{U}(\beta) \neq \varnothing$. Therefore, if we have a collection S of $k \geq 1$ combinatorially distinct train track maps on r-roses representing fully irreducible automorphisms of F_r and if $m \geq 1$ is such that each $f \in S$ represents $\varphi_f \in \mathrm{Out}(F_r)$ with $\#\mathbf{U}(\varphi_f) \leq m$, then the collection $\{[\varphi_f] | f \in S\}$ contains $\geq k/m$ distinct $\mathrm{Out}(F_r)$-conjugacy classes of fully irreducibles.*

For a train track map $f : \Gamma \to \Gamma$ define $||f|| := \sum_{e \in E\Gamma} |f(e)|$, where the summation is taken over all topological edges of Γ, and where $|f(e)|$ is the combinatorial length of the path $f(e)$.

Lemma 5.4. *Let $f : R_r \to R_r$ be a train track map representing an ageometric lone axis fully irreducible $\varphi \in Out(F_r)$, where $r \geq 3$. Then*

$$\#\mathbf{U}(\varphi) \leq ||f||.$$

Proof. Choose in the $Out(F_r)$ conjugacy class of φ the outer automorphism $\varphi' \in Out(F_r)$ that has a train track representative g on the rose R_r with the identity marking. Thus $g = f$ as a map on the rose R_r, and the only difference between g and f is in modifying the marking.

Since having a lone axis is a conjugacy class invariant, φ' also has a unique axis $\mathcal{A}_{\varphi'}$. By Theorem 3.6, $\mathcal{A}_{\varphi'}$ is the periodic fold line obtained from each train track representative of φ'. Call by σ the segment of $\mathcal{A}_{\varphi'}$ starting at R_r with the identity marking and consisting of a single Stallings fold decomposition of g (a single period of the periodic fold line for g). The lone axis property of φ' implies that all elements of $\mathbf{U}(\varphi')$ arise from the r-roses that occur along σ.

Using Remark 3.3, it is not difficult to see that a period consists of $\frac{||g||}{2}$ folds. Hence, σ passes through at most $\frac{||g||}{2} + 1$ roses R_r. (Note that in the middle of a fold the underlying graph always has a trivalent vertex and is therefore never the r-rose R_r). Therefore, the element $\varphi' \in \mathrm{Out}(F_r)$ has at most $\frac{||g||}{2}$ unmarked train track representatives based on the rose R_r, as does its conjugate φ. Since as unmarked graph maps $g = f$, we have $||g|| = ||f||$. Hence $\#\mathbf{U}(\varphi) \leq ||f||$, as claimed. ☐

Theorem 5.5. *Let $r \geq 3$. Then there exist constants $c = c(r) > 1$ and $L_0 \geq 1$ such that for each $L \geq L_0$ the number $\mathfrak{NA}_r(L)$ of distinct $Out(F_r)$-conjugacy classes of ageometric lone axis fully irreducibles $\varphi \in Out(F_r)$ with $\log \lambda(\varphi) \leq L$ satisfies*

$$\mathfrak{NA}_r(L) \geq c^{e^L}.$$

Therefore, c^{e^L} bounds below the number of closed geodesics in $\mathcal{M}_r$ of length bounded above by L.

Proof. Let $L \geq 1$. Let $Z_+(L)$ denote the set of all full positive words of length e^L in $x_2, \ldots, x_r$.

By the law of large numbers, the probability that a uniformly at random chosen positive word in $x_2, \ldots, x_r$ of length n is full tends to 1 as $n \to \infty$. Therefore

$$\lim_{n \to \infty} \frac{\#\{w \mid w \text{ is a full positive word in } x_2, \ldots, x_r \text{ of length } n\}}{(r-1)^n} = 1$$

and so $\lim_{L \to \infty} \frac{\#Z_+(L)}{(r-1)^{e^L}} = 1$. In particular, there exists a sufficiently large $L_0' \geq 1$ such that for all $L \geq L_0'$ we have $\frac{\#Z_+(L)}{(r-1)^{e^L}} \geq 1/2$, that is $\#Z_+(L) \geq (r-1)^{e^L}/2$.

For each such word $z \in Z_+(L)$, the word $w = x_{r-1} z x_2$ is also a full positive word and begins in x_{r-1} and ends in x_2. Define $W_+(L) := \{x_{r-1} z x_2 \mid z \in Z_+(L)\}$. Thus $\#W_+(L) \geq (r-1)^{e^L}/2$ for each $L \geq L_0$, and for each $w \in W_+(L)$ we have $|w| = e^L + 2$.

For each $w \in W_+(L)$ consider the train track map $f_w = g_w \circ g_{12,1}$ as in Proposition 4.9 above. Proposition 4.9 implies f_w represents an ageometric lone axis fully irreducible $\varphi_w \in Out(F_r)$. We claim that for $w, w' \in W_+(L)$, we have $\varphi_w = \varphi_{w'}$ if and only if $w = w'$. Indeed, suppose, on the contrary, that $w, w' \in W_+(L)$ are distinct words, but that $\varphi_w = \varphi_{w'}$. Denote by ψ_w, $\psi_{w'}$, and β

the elements of $\mathrm{Out}(F_r)$ represented by $g_w, g_{w'}$, and $g_{12,1}$ respectively. We have $\varphi_w = \varphi_{w'} = \psi_w \beta = \psi_{w'}\beta$ and therefore $\psi_w = \psi_{w'} = \varphi_w \beta^{-1} = \varphi_{w'}\beta^{-1}$. By definition, $\psi_w([x_r]) = [x_1 w(x_2, \dots, x_r)]$ and $\psi_{w'}([x_r]) = [x_1 w'(x_2, \dots, x_r)]$. (Here for $u \in F_r$ we denote by $[u]$ the conjugacy class of u in F_r). Since by assumption $w \neq w'$ are distinct positive words in $x_2, \dots, x_r$, the words $x_1 w(x_2, \dots, x_r)$ and $x_1 w'(x_2, \dots, x_r)$ are distinct positive words in $x_1, \dots, x_r$, which are cyclically reduced and therefore not conjugate in $F_r = F(x_1, \dots, x_r)$. This contradicts the assumption that $\varphi_w = \varphi_{w'}$. Thus the claim is verified, and we know that distinct words $w \in W_+(L)$ define distinct outer automorphisms $\varphi_w \in Out(F_r)$. Since for each $w \in W_+(L)$ we have that $f_w : R_r \to R_r$ is a train track representative of φ_w, it follows that distinct words in $w \in W_+(L)$ produce distinct marked train track maps $f_w : R_r \to R_r$, where R_r is taken with the identity marking. Two such maps f_w and $f_{w'}$ can still be equivalent, in the sense of Definition 5.1, if they are conjugate by a graph automorphism of R_r. There are $m = 2^r r!$ simplicial automorphisms of R_r. We thus have, for each $L \geq L_0'$, a collection $\{f_w : R_r \to R_r | w \in W_+(L)\}$ of at least $(r-1)^{e^L}/(2m)$ combinatorially distinct (in the sense of Definition 5.1) train track maps.

For $w \in W_+(L)$ we have $||g_w|| = r + |w| = r + 2 + e^L$. Since $g_{12,1}$ is fixed and does not depend on w or L, there exists a constant $K = K(r) \geq 1$ such that $||f_w|| \leq Ke^L$. By Lemma 5.4, for each $w \in W_+(L)$, we have $\#\mathbf{U}(\varphi_w) \leq ||f_w|| \leq Ke^L$.

Then, by Remark 5.3, for $L \geq L_0'$, the number of distinct $Out(F_r)$ conjugacy classes represented by $\{f_w | w \in W_+(L)\}$ is

$$\geq \frac{(r-1)^{e^L}}{2mKe^L} \geq_{L\to\infty} (r-1.5)^{e^L}.$$

For each $w \in W_+(L)$ we have $||f_w|| \leq Ke^L$, and therefore $\lambda(f_w) \leq ||f_w|| \leq Ke^L$. (Here we are using the fact that the Perron-Frobenius eigenvalue $\lambda(f_w)$ is bounded above by the maximum of the row-sums of the transition matrix $M(f_w)$; see [Sen73] for details.) Hence

$$\log \lambda(\varphi_w) = \log \lambda(f_w) \leq L + \log K.$$

Now let $L_1 = L + \log K$. Then from above we have $\log \lambda(\varphi_w) = \log \lambda(f_w) \leq L + \log K = L_1$. Also, the number of distinct $Out(F_r)$ conjugacy classes represented by $\{f_w | w \in W_+(L)\}$ is

$$\geq_{L\to\infty} (r-1.5)^{e^L} = (r-1.5)^{e^{L_1 - \log K}} = \left((r-1.5)^{1/K}\right)^{e^{L_1}},$$

which completes the proof of the main statement.

The final sentence of the theorem follows from the fact that the translation distance of φ along A_φ is $log(\lambda(\varphi))$. $\square$

Lemma 5.6. *Let $r \geq 2$. Then there exist $a > 1, b > 1$ such that for each $L \geq 1$ the number of expanding irreducible train track maps $f : \Gamma \to \Gamma$, where $b_1(\Gamma) = r$ and $\log \lambda(f) \leq L$, is $\leq a^{b^L}$.*

Proof. This proof follows the argument in [HK17, Remark 3.3]. First note that there are only finitely many choices for a finite connected graph Γ where all vertices have degree ≥ 3 and $b_1(\Gamma) = r$. Thus we may assume Γ is fixed. Let $k = \#E\Gamma$ and let $M = (m_{ij})_{ij=1}^k$ be $M(f)$. By [BK16, Proposition A.4], if f is as above and $\lambda := \lambda(f)$, then $\max m_{ij} \leq k\lambda^{k+1}$. If $\log \lambda \leq L$, we get $\max \log m_{ij} \leq \log k + (k+1)L$ and $\max m_{ij} \leq ke^{(k+1)L}$. Thus we get exponentially many (in terms of L) possibilities for transition matrices $M(f)$. Since for a given length s there are exponentially many paths of length s in Γ, we get a double exponential upper bound for the number of expanding irreducible train track maps $f : \Gamma \to \Gamma$ with $\log \lambda(f) \leq L$, as required. $\square$

Remark 5.7. *The Lemma 5.6 proof shows that if $f : \Gamma \to \Gamma$ is an expanding irreducible train track map representing some $\varphi \in \mathrm{Out}(F_r)$ with $\log \lambda(\varphi) = \log \lambda(f) \leq L$, then for the coefficients m_{ij} of $M(f)$ we have $\max m_{ij} \leq ke^{(k+1)L}$, where $k = \#E\Gamma$. For a fixed $r \geq 2$, the number of possible Γ is finite (since $\pi_1(\Gamma) \cong F_r$ and each vertex of Γ has degree ≥ 3), so the number k of rows/columns of*

$M(f)$ is bounded in terms of r. And for $L \geq 1$ we have an exponential upper bound h^L (for some $h = h(r) > 1$) on the number of possible transition matrices $M(f)$. Thus, for $L \geq 1$,

$$\#\{\lambda(\varphi) | \varphi \in \mathrm{Out}(F_r) \text{ is fully irreducible with } \log \lambda(\varphi) \leq L\} \leq h^L.$$

This fact provides stark contrast with the double exponential lower bound given by Theorem 5.5.

Theorem 5.8. *For each integer $r \geq 3$, there exist constants $a = a(r) > 1, b = b(r) > 1, c = c(r) > 1$ so that: For $L \geq 1$, let $\mathfrak{N}_r(L)$ denote the number of $Out(F_r)$-conjugacy classes of fully irreducibles $\varphi \in Out(F_r)$ with $\log \lambda(\varphi) \leq L$. Then there exists an $L_0 \geq 1$ such that for each $L \geq L_0$ we have*

$$c^{e^L} \leq \mathfrak{N}_r(L) \leq a^{b^L}.$$

Therefore, c^{e^L} bounds below the number of closed geodesics in $\mathcal{M}_r$ of length bounded above by L.

Proof. The lower bound follows directly from Theorem 5.5. Since every fully irreducible $\varphi \in Out(F_r)$ can be represented by an expanding irreducible train track map $f : \Gamma \to \Gamma$ with $\lambda(\varphi) = \lambda(f)$, the upper bound follows from Lemma 5.6. □

6. Questions

Define a function $\omega : [0, \infty) \to [0, \infty)$ to have *double exponential asymptotics* if there exist numbers $a > 1, b > 1, c > 1, d > 1$ and $t_0 \geq 1$ such that for all $t \geq t_0$,

$$c^{d^t} \leq \omega(t) \leq a^{b^t}.$$

Describing the precise asymptotics of such functions appears to be a nontrivial analytic problem. As an initial approach, for a function $\omega(t)$ with double exponential asymptotics we define the *principal entropy* $b = b(\omega) > 1$ as

$$(\dagger) \qquad \log b := \limsup_{t \to \infty} \frac{\log \log \omega(t)}{t}.$$

Note that if $\omega(t) = a^{b^t}$, for constants $a > 1, b > 1$, then the principal entropy of ω is exactly b.

Now, if $\omega(t)$ is a function with double exponential asymptotics and with principal entropy $b = b(\omega)$, we define the *secondary entropy* $a = a(\omega)$ as

$$\log a := \limsup_{t \to \infty} \frac{\log \omega(\log_b t)}{t}.$$

Again, if $\omega(t) = a^{b^t}$, where $a > 1, b > 1$ are constants, then the secondary entropy of ω is exactly a.

Recall that $\mathfrak{N}_r(L)$ is the number of $Out(F_r)$-conjugacy classes of fully irreducibles $\varphi \in Out(F_r)$ with $\log \lambda(\varphi) \leq L$.

Question 6.1. *Let $r \geq 3$ and $\omega(L) = \mathfrak{N}_r(L)$.*

1. *What is the principal entropy of $a(\omega)$?*
2. *Does $a(\omega)$ depend on r?*
3. *Is it true that $a(\omega) = e$? (Theorem 5.5 does imply that $a(\omega) \geq e$.)*
4. *Does the actual limit exists for ω in $(\dagger)$ in this case?*
5. *What is the secondary entropy $b(\omega)$? Does it depend on r and how?*

References

[AKKP17] Y. Algom-Kfir, I. Kapovich, and C. Pfaff. Stable strata of geodesics in outer space. *International Mathematics Research Notices, to appear*, 2017.

[BF14] M. Bestvina and M. Feighn. Hyperbolicity of the complex of free factors. *Advances in Mathematics*, 256:104–155, 2014.

[BH92] M. Bestvina and M. Handel. Train tracks and automorphisms of free groups. *The Annals of Mathematics*, 135(1):1–51, 1992.

[BK16] M. Bell and I. Kapovich. Detecting fully irreducible automorphisms: a polynomial time algorithm. *Experimental Mathematics, to appear*, 2016.

[Bog08] O. Bogopolski. *Introduction to group theory*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. Translated, revised and expanded from the 2002 Russian original.

[BSV14] Mladen Bestvina, Michah Sageev, and Karen Vogtmann. *Geometric Group Theory*, volume 21. American Mathematical Society, 2014.

[Bus92] P. Buser. *Geometry and spectra of compact Riemann surfaces*, volume 106 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1992.

[CV86] M. Culler and K. Vogtmann. Moduli of graphs and automorphisms of free groups. *Inventiones mathematicae*, 84(1):91–119, 1986.

[DKL15] S Dowdall, I. Kapovich, and C. J. Leininger. Dynamics on free-by-cyclic groups. *Geom. Topol.*, 19(5):2801–2899, 2015.

[DT15] Spencer Dowdall and Samuel Taylor. Contracting orbits in Outer space. *arXiv preprint arXiv:1502.04053*, 2015.

[EM11] A. Eskin and M. Mirzakhani. Counting closed geodesics in moduli space. *J. Mod. Dyn.*, 5(1):71–105, 2011.

[FH11] M. Feighn and M. Handel. The recognition theorem for $\mathrm{Out}(F_n)$. *Groups Geom. Dyn.*, 5(1):39–106, 2011.

[HK17] M. Hull and I. Kapovich. Counting conjugacy classes in $Out(F_N)$. *Bulletin of the Australian Mathematical Society, to appear*, 2017.

[HM11] M. Handel and L. Mosher. *Axes in outer space*. Number 1004. Amer Mathematical Society, 2011.

[Kap14] I. Kapovich. Algorithmic detectability of iwip automorphisms. *Bulletin of the London Mathematical Society*, 46(2):279–290, 2014.

[KP15] I. Kapovich and C. Pfaff. A train track directed random walk on $\mathrm{Out}(F_r)$. *International Journal of Algebra and Computation*, 25(5):745–798, August 2015.

[Mar69] G. A. Margulis. Certain applications of ergodic theory to the investigation of manifolds of negative curvature. *Funkcional. Anal. i Priložen.*, 3(4):89–90, 1969.

[Mar70] G. A. Margulis. *On some aspects of the theory of Anosov flows*. PhD thesis, Moscow State University, 1970.

[MP16] L. Mosher and C. Pfaff. Lone axes in outer space. *Algebr. Geom. Topol.*, 16(6):3385–3418, 2016.

[Pfa12] C. Pfaff. *Constructing and Classifying Fully Irreducible Outer Automorphisms of Free Groups*. PhD thesis, Rutgers University, 2012.

[Pfa13] C. Pfaff. Ideal Whitehead graphs in $\mathrm{Out}(F_r)$ II: the complete graph in each rank. *Journal of Homotopy and Related Structures*, 10(2):275–301, 2013.

[Pfa15] C. Pfaff. $Out(F_3)$ index realization. *Math. Proc. Cambridge Philos. Soc.*, 159(3):445–458, 2015.

[Sen73] E. Seneta. *Non-negative matrices*. Halsted Press [A division of John Wiley & Sons], New York, 1973. An introduction to theory and applications.

[Sko89] R. Skora. Deformations of length functions in groups, preprint. *Columbia University*, 1989.

[Sta83] J.R. Stallings. Topology of finite graphs. *Inventiones Mathematicae*, 71(3):551–565, 1983.

Department of Mathematics, University of Illinois at Urbana-Champaign
1409 West Green Street, Urbana, IL 61801
https://faculty.math.illinois.edu/~kapovich,
E-mail address: kapovich@math.uiuc.edu

Department of Mathematics, University of California at Santa Barbara
South Hall, Room 6607; Santa Barbara, CA 93106-3080
http://math.ucsb.edu/~cpfaff/,
E-mail address: cpfaff@math.ucsb.edu