A Nielsen-Thurston inspired story of iterating free group automorphisms and efficiently deforming graphs

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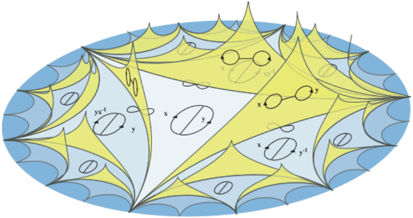
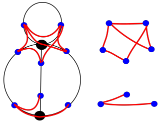


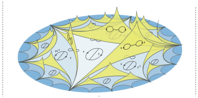
Is A Tale With Two Surprisingly Interconnected Themes

What happens when you iterate a free group automorphism?	⇜⇝	What happens when you efficiently deform a metric graph?

- Outer automorphism invariants



- Geodesics in Culler-Vogtmann Outer Space



Main Character: Outer Automorphism Group of the Free Group $Out(F_r)$

$F_r = < x_1, \ldots, x_r >$ rank r free group

Definition

$Out(F_r) = \frac{Aut(F_r)}{Inn(F_r) = \{\varphi_a | \ \varphi_a(b) = aba^{-1} \ \forall \ a, b \in F_r\}}$

To define $\Phi \in Aut(F_r)$, just need to describe images of generators:

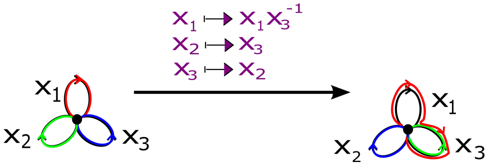
$$\Phi = \begin{cases} x_1 \mapsto x_1 x_3^{-1} \\ x_2 \mapsto x_3 \\ x_3 \mapsto x_2 \end{cases}$$

Outer Automorphism Group of the Free Group $Out(F_r)$

To utilize work of Nielsen, Skora, Stallings, Whitehead, and Bestvina-Feighn-Handel we view

$$\Phi = \begin{cases} x_1 \mapsto x_1 x_3^{-1} \\ x_2 \mapsto x_3 \\ x_3 \mapsto x_2 \end{cases}$$

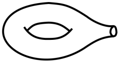
as a homotopy equivalence of graphs:



Definition

$\varphi \in Out(F_r)$ is *fully irreducible (f.i.)* if no positive power φ^k fixes the conjugacy class of a proper free factor of F_r.

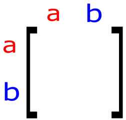
The Backstory

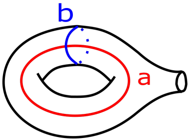
$$GL_2(Z) \cong MCG($$  $$) \cong Out(F_2)$$

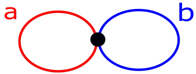
2x2 integer matrices
of determinant +/- 1

Homeo() / Homotopy

$Aut(F_2) / Inn(F_2)$

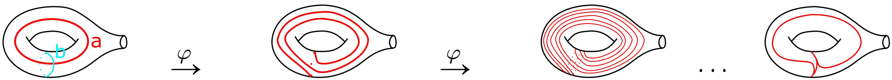






Nielsen-Thurston studied asymptotic dynamical invariants

For φ a generic (pseudo-Anosov) surface homeo, repeated application of φ to any curve limits on the same object...



$$\begin{bmatrix}1\\0\end{bmatrix} \xrightarrow{\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}} \begin{bmatrix}2\\1\end{bmatrix} \xrightarrow{\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}} \begin{bmatrix}5\\3\end{bmatrix} \quad \cdots \quad k\begin{bmatrix}\frac{1+\sqrt{5}}{2}\\1\end{bmatrix}$$

Some important "conjugacy class" invariants:

$GL(2,\mathbb{Z})$	$MCG(\Sigma_{1,1})$	$Out(F_2)$
• Eigenvector	• Lamination*	• Lamination
		• Axis bundle
	• Indices / IWG	• Indices / IWG

* Connected to "measured foliations"

$\mathcal{T}(S)$: Deformation space of hyp. metrics on a surface S

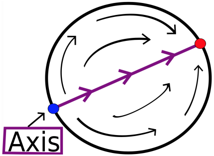
• Amazingly, the space of metrics is itself a metric space &...

• [Royden] For closed surface S: $Isom(\mathcal{T}(S)) \cong MCG(S)$

• [Thurston] $\mathcal{T}(S)$ is compactified by projective measured foliations on S & $\overline{\mathcal{T}(S)}$ is a ball

• Bers, Marden, Masur, Strebel, Thurston, & others connected mapping class group invariants with geodesics in $\mathcal{T}(S)$

$\varphi \circlearrowright$



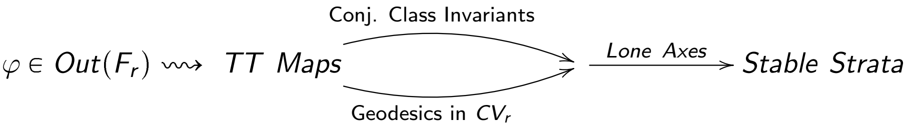
$\overline{\mathcal{T}(S)}$

The $Out(F_r)$ Tale: Interconnected Goals & Strategy

Interconnected Goals:

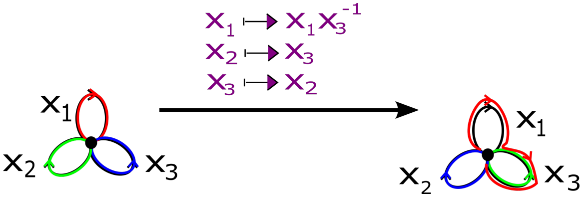
1. Understanding generic $Out(F_r)$ conjugacy class invariants
2. Understanding Geodesics in Outer Space CV_r

Strategy/Outline:



Train Track Representatives (Bestvina-Handel)

Recall: $\varphi \in Out(F_r)$ always have topological representatives:

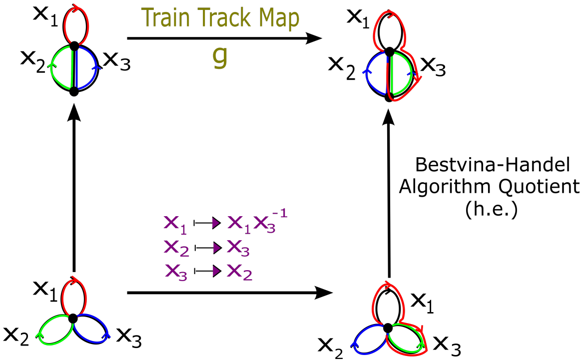


- But iteration may lead to cancellation on edge interiors

Train Track Representatives (Bestvina-Handel)

f.i. $\varphi \in Out(F_r)$ have **train track representatives** $g\colon \Gamma \xrightarrow{h.e.} \Gamma$

- $g_*\colon \pi_1(\Gamma) \to \pi_1(\Gamma)$ is φ
- $g^k \mid_{int(e)}$ is locally injective $\forall$ edges e of Γ, $k > 0$

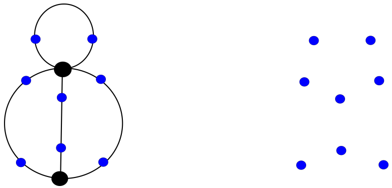


$\mathcal{IW}(\varphi)$: An $Out(F_r)$ Conjugacy Class Invariant

Idea in analogy with Nielsen-Thurston setting:

- Iterate loops $\rightsquigarrow$ Lamination leaves
- Record at vertices how lamination leaves enter & leave

Combinatorially:



1. Vertex for each directed edge at each node

$\mathcal{IW}(\varphi)$: An $Out(F_r)$ Conjugacy Class Invariant

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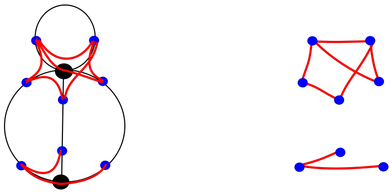


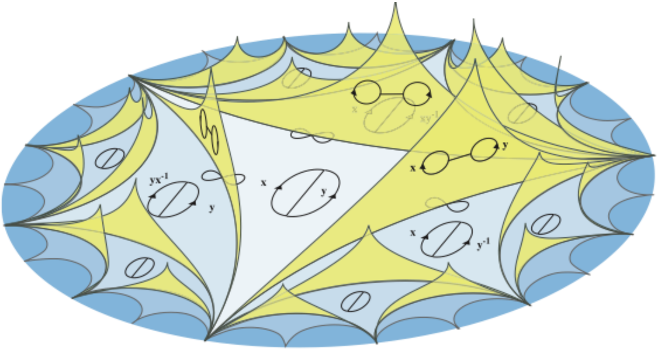
1. Vertex for each directed edge at each node
2. Take any edge
3. Look at its image after applying g iteratively

Strategy/Outline:

$\varphi \in Out(F_r) \rightsquigarrow$ *TT Maps* $\xrightarrow{\text{Conj. Class Invariants}}$ / $\xrightarrow{\text{Geodesics in } CV_r}$ $\xrightarrow{\textit{Lone Axes}}$ *Stable Strata*

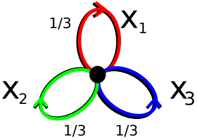
$Out(F_r)$ & the Deformation Space of Metric Graphs

$Out(F_r)$ is the isometry group for a deformation space of metric graphs, Culler-Vogtmann Outer Space CV_r

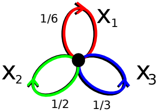


Points in the Outer Space CV_r

Points in CV_r are *marked*, *metric*, *graphs*:

Most basic point: 

Can additionally:

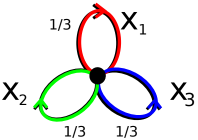


Change lengths
on edges

Outer Space CV_r

Points in CV_r are *marked*, *metric*, *graphs*:

Most basic point:



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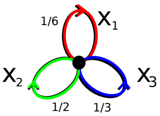
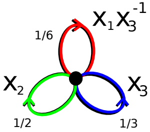


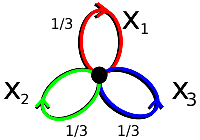
Change lengths on edges



Apply automorphism

Outer Space CV_r

Points in CV_r are *marked*, *metric*, *graphs*:

Most basic point: 

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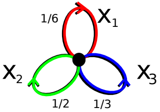
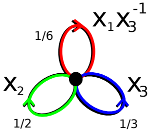
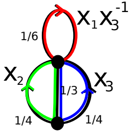


Change lengths on edges



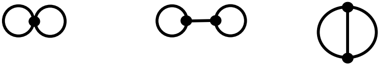
Apply automorphism



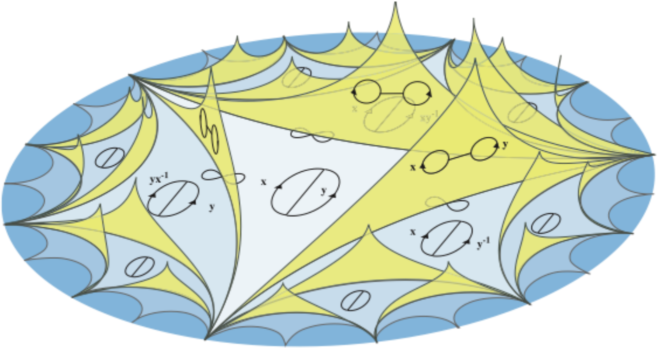
"Blow up" vertex

Outer Space in Rank 2 (CV_2)

The graphs Γ with $\pi_1(\Gamma) = F_2$:

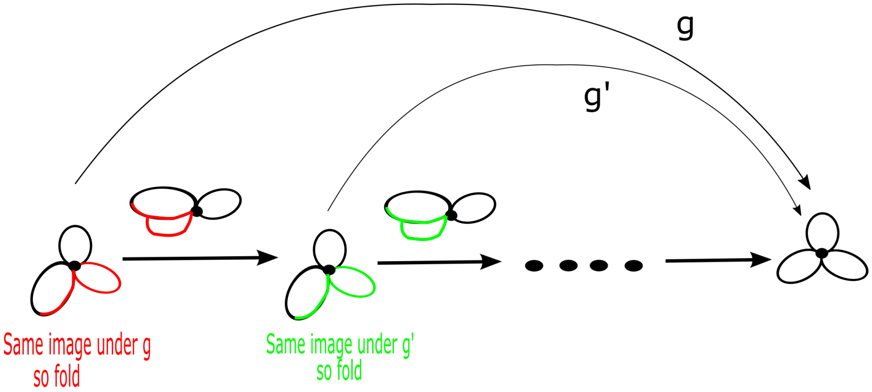


CV_2:



TT Maps $g \rightsquigarrow$ Geodesics in CV_r

- Stallings allows us to decompose a tt map as a sequence of "folds"
- Skora made continuous, to define geodesics in CV_r



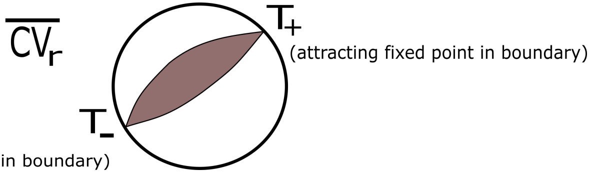
Strategy/Outline:

$$\varphi \in Out(F_r) \rightsquigarrow TT\ Maps \xrightarrow[\text{Geodesics in } CV_r]{\text{Conj. Class Invariants}} \xrightarrow{Lone\ Axes} Stable\ Strata$$

Surprising Connection: Lone Axes in Outer Space

Definition (Axis bundle $\mathcal{A}_\varphi$ for nice $\varphi \in Out(F_r)$ (Handel, Mosher))

$\mathcal{A}_\varphi = \overline{\{\text{Fold line geodesics for tt reps of } \varphi^k \textit{ with } k > 0\}} \subset CV_r$



Theorem (Main Theorem I; Mosher, Pfaff)

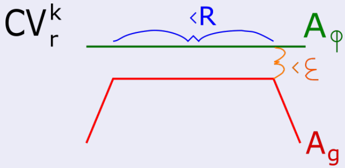
(Algorithmically checkable) $\mathcal{IW}(\varphi)$ condition for when $\mathcal{A}_\varphi$ a "lone axis"

- Exist by Pfaff, many by I. Kapovich-Pfaff,
- different kinds by Coulbois-Lustig

Test for generic behavior (Stable Strata)

Theorem (Main Theorem II; Algom-Kfir, I. Kapovich, Pfaff)

$\varphi \in Out(F_r)$ lone axis f.i. st all components of $\mathcal{IW}(\varphi)$ are complete gphs $\Longrightarrow$ $\exists$ constants R,ε st if $\exists$ a tt map g with an axis A_g satisfying



- *Then g represents an ageometric f.i. ψ st either*
  1. $\mathcal{IW}(\psi) \cong \mathcal{IW}(\varphi)$ *or*
  2. $\mathcal{IW}(\psi) \ncong \mathcal{IW}(\varphi)$ *& pathology occurs*
- *(2) can happen*
- *(2) cannot happen if $\mathcal{IW}(\varphi)$ is the complete gph on $2r-1$ vertices*

Unhatched Egg: Random Walks

- I. Kapovich-Pfaff proved complete gph on $2r - 1$ vertices generic as $\mathcal{IW}(\varphi)$ along "tt directed" random walk
- Gadre-Maher used "stable strata" theorems to prove generic pseudo-Anosov index list

Question (Work in progress with Algom-Kfir, I. Kapovich, J. Maher)

- Does Main Theorem II indicate that complete ideal Whitehead graphs are generic for $Out(F_r)$?
- Or other stable strata graphs?

Thank you!