Handout Problems 3/7/06

Problem 1. Let $P, Q \in \mathbb{R}^n$ and $\gamma : [0, 1] \to \mathbb{R}^n$ satisfy $\gamma(0) = P$, $\gamma(1) = Q$. Show that

$$\left\| P - Q \right\| \leq \int_0^1 \left\| \gamma'(t) \right\| dt$$

and that equality holds iff $\gamma(t) = P + g(t)(Q - P)$ where $g : [0, 1] \to [0, 1]$ is nondecreasing and $g(0) = 0$, $g(1) = 1$. You might find the triangle inequality for $\left\| \cdot \right\|$ and the definition of the integral useful for the first part.

Problem 2. We first state a form of the implicit function theorem (imft) and then ask you to use it at the end. Let

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_m)$$

denote points in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let

$$f(x, y) = (f_1(x, y), f_2(x, y), \ldots, f_n(x, y)) \in \mathbb{R}^n$$

be defined for

$$x \in B_R(x^0) = \{ x \in \mathbb{R}^n : \| x - x^0 \| < R \}, \quad y \in B_r(y^0) = \{ y \in \mathbb{R}^m : \| y - y^0 \| < r \}$$

where $(x^0, y^0) \in \mathbb{R}^n \times \mathbb{R}^m$ and $0 < R, r$ are given. Assume that $f$ is continuous in $(x, y)$ together with its Jacobian with respect to $x$:

$$Jf = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}.$$  

Assume that

$$f(x^0, y^0) = 0$$

is invertible.

Under these conditions, there exist $0 < R_0 < R$, $0 < r_0 < r$ such that:

If $\| y - y^0 \| < r_0$, then $f(x, y) = f(x^0, y^0)$ has a unique solution $x$

which satisfies $\| x - x^0 \| < R_0$. Moreover, if $x = g(y)$ is this solution, then $g : B_{R_0}(y^0) \to B_{R_0}(x^0)$ is continuous. Moreover, if $f$ has continuous partial derivatives of order at most $k$ in all variables $x, y$,

then the partial derivatives of $g$ of order at most $k$ are also exist and are continuous.

Let $A^0$ be an $n \times n$ matrix, $x^0 \in \mathbb{R}^n$ be a unit vector. Assume that $0$ is a simple root of $p(\lambda) = \det(A - \lambda I)$ and

$$A^0 x^0 = 0.$$  

That is, $0$ is a simple eigenvalue of $A^0$ and $x^0$ is a corresponding (unit) eigenvector. Show that then if $A$ is near $A^0$ then $A$ has a unique eigenvalue $\lambda$ near $0$ and corresponding eigenvector $x = x^0 + z$ where $x^0 \cdot z = 0$ and $z$ is near $0$. Set it up like this: define

$$f(A, \lambda, z) = (A - \lambda I)(x^0 + z)$$

and use the imft on

$$f(A, \lambda, z) = f(A^0, 0, 0) = 0$$

to find $\lambda, z$ as smooth functions of $A$ near $A^0$. For simplicity, use coordinates in which

$$x^0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

so that $z = \begin{bmatrix} z_1 \\ \vdots \\ z_{n-1} \\ 0 \end{bmatrix}$.

Problem 3. Let $\sigma$ be a patch in a surface $S$ with the FFF $du^2 + \cosh^2(u)dv^2$. Let $\tilde{\sigma}$ be given by $\tilde{\sigma}(V, W) = \sigma(u, v)$ when

$$V = e^t \tanh(u), \quad W = e^t \sech(u).$$

Show the FFF of $\tilde{\sigma}$ is $\frac{4 \cosh^2(u)}{e^{2t}}$. You may use Problem 5.4 of the text if you like.