Arch. Hist. Exact Sci. 56 (2001) 39–68 © Springer-Verlag 2001

John von Neumann's Conception of the Minimax Theorem: A Journey Through Different Mathematical Contexts

TINNE HOFF KJELDSEN

Communicated by J. GRAY

1. Introduction

The first purpose of this paper is to tell the history of John von Neumann's development of the minimax theorem for two-person zero-sum games from his first proof of the theorem in 1928 until 1944 when he gave a completely different proof in the first coherent book on game theory. I will argue that von Neumann's conception of this theorem as a theorem belonging to the theory of linear inequalities as well as his awareness of its connection to fixed point theorems were absent in 1928. In contradiction to the impression given in the literature these connections were only gradually recognized by von Neumann over time. By reading this knowledge into von Neumann's first proof of the minimax theorem from 1928 a major part of the cognitive development of this theorem is neglected within the history of mathematics. The significance of interactions between different branches of mathematics for the conception and development of the minimax theorem are neglected as well. This paper will remedy this and shed new light on these issues.

Since the beginning of the nineties there has been an increasing interest in the history of game theory, several historical papers have appeared and most of them of course mention von Neumann's 1928 proof of the minimax theorem. A common feature though is that none of these give an analysis of the mathematics in von Neumann's proof. There is only one paper that goes deeper into the mathematics. It is an old essay written by two Princeton mathematicians, the late Albert W. Tucker and Harold W. Kuhn, in memory of John von Neumann. They treat the mathematics in a modern (1958) framework and emphasize in particular the connections to fixed point theorems and the theory of linear inequalities [Kuhn and Tucker, 1958, p. 111–112]. The other historical papers say little more about von Neumann's 1928 proof of the minimax theorem than that it is very difficult.[1] Von Neumann's biographer Steve J. Heims very tellingly called it "a tour de force" [Heims, 1980, p. 91]. Some of the papers also state that the proof is about

[1] See [Dimand and Dimand, 1992, p. 24], [Leonard, 1992, p. 44], [Ingrao and Israel, 1990, p. 211].

systems of linear inequalities and equations[2] and one claims that it is based on fixed point theorems[3]. Reading von Neumann's 1928 paper I found that these statements were not at all obvious; as a matter of fact von Neumann did not talk about fixed points in his 1928 proof and he did not formulate or present a system of linear inequalities and equations to be solved. Today we know that all these connections are there but von Neumann doesn't seem to have been fully aware of this in 1928, rather it was an insight that emerged gradually during his work from 1928 until 1944 in which the minimax theorem – sometimes surprisingly – presented itself in different mathematical contexts.

I will argue for this claim of gradually emerging insight through an analysis of von Neumann's 1928-paper, of a paper he published in 1937 on a mathematical model for an expanding economy, and of the proof of the minimax theorem that appeared in von Neumann and Morgenstern's famous book on game theory published in 1944.

The second purpose of this paper is to discuss a more philosophical issue concerning the significance of the context in which a theorem is developed. The importance of a mathematical theorem is dependent on the branch or discipline of mathematics within which it is considered. A mathematical result is not likely to be deemed equally important within different branches or contexts of mathematics. The interesting questions, the questions that guide the research in different mathematical contexts are not the same. Thus, the potential of a mathematical theorem for stimulating further research is dependent of the mathematical context of discovery.[4]

The background for these questions in relation to the history of the minimax theorem is a dispute in 1953 between von Neumann and the French mathematician Maurice Fréchet about who should be named the initiator of game theory – an honour the mathematical literature at that time associated with von Neumann. Fréchet argued that even though Émile Borel was not able to prove the minimax theorem he was the true initiator of game theory because of his treatment of the subject in papers published at the beginning of the twenties, before von Neumann's 1928 paper. The interesting issue is not to settle the priority between Borel and von Neumann but rather to analyze the significance of the minimax theorem. According to Fréchet the minimax theorem was not such an important result because it turned out that it can be derived very easily from other theorems on linear inequalities, theorems proved before 1928. The underlying assumption behind Fréchet's argument is that theorems that turn out to be equivalent have the same significance or the same potential for stimulating further mathematical developments regardless of the mathematical context in which they were derived. This touches the philosophical issue raised above. A contextualized analysis of similar mathematical theorems derived in different mathematical contexts can give the historian a

---

[2] See [Ingrao and Israel, 1990, p. 211], [Heims, 1980, p. 91].

[3] See [Ingrao and Israel, 1990, p. 211].

[4] For example both the calculus of variations and mathematical programming treat optimization under constraints. In the calculus of variations the infinite cases are treated whereas mathematical programming is concerned with finite dimensional cases; so a theorem about constrained optimization can be deemed very important and can lead to new knowledge in mathematical programming whereas the same theorem evaluated from the point of view of the calculus of variations is seen as just a minor thing. For a specific case see footnote 5.

tool for understanding processes behind the divison of mathematical results that gave rise to new developments in mathematics and results that did not.[5]

This issue is discussed in the second part[6] of the paper on the basis of an analysis of some of Borel's papers on game theory, of von Neumann's work, and of the dispute between Fréchet and von Neumann.

2. The first proof of the minimax theorem: von Neumann's 1928 paper

John (Johann) von Neumann (1903–1957) published his first paper on what he called "Theorie der Gesellschaftsspiele" in 1928. From there on it took 16 years before he published on game theory again. This has of course given rise to some speculations about why and where the idea and the inspiration to develop a mathematical theory of games came from. Two explanations have been suggested in the literature, one of them is that von Neumann got the idea from reading Borel's work on the subject.[7] As to this explanation von Neumann himself claimed in a short note presented by Borel May 14, 1928[8] at *l'Académie des Sciences*, shortly before the publication of his paper, that he developed the theory independently of Borel [von Neumann, 1928a]. He explicitly stated so in a footnote in the 1928 paper[9] where he tells that someone drew his attention to Borel's notes during the proofreading, indicating that he had no knowledge of Borel's work until he had completed his own work [von Neumann, 1928, p. 306]. Von Neumann's claim is supported by the course of events: Even though he presented his work at the weekly seminar of the mathematical institute in Göttingen in December 1926 [von Neumann,

---

[5] The Kuhn-Tucker theorem in nonlinear programming is an example of this. Kuhn and Tucker derived the theorem in 1950 and it imediately launched the theory of nonlinear programming and became viewed as a very important result. Later it turned out that a similar result had been proven 11 years earlier by William Karush in his master thesis. Karush's work was done in the mathematical context of the calculus of variations within which is was not regarded as a very important result, and was not even published. Fritz John also proved a similar result, but he encountered problems in getting it published. It finally appeared in print in 1948 – only two years before Kuhn and Tucker's version of the theorem was published. John's work was done in the context of the theory of convexity, in which the theorem was once again not deemed to be some thing special. The reasons for the very different receptions of these results within the mathematical community can be explained by referring to the significance of the different mathematical contexts in which the results were derived. For an analysis of this see [Kjeldsen, 2000b].

[6] The second part of the paper starts at Sect. 6. The discussions and conclusions in that part draw on the analysis of von Neumann's work presented in the first part of the paper, which means that the second part of the paper cannot stand alone.

[7] See for example [Ulam, 1958, p. 7].

[8] The note was published in *Comptes Rendus de l'Académie des Sciences* June 18, 1928 [von Neumann, 1928a].

[9] There are two publications by von Neumann from 1928. The first one is labelled [von Neumann, 1928a] and refers to the short note in *Comptes Rendus de l'Académie des Sciences*. The second one is labelled [von Neumann, 1928] and refers to von Neumann's paper "Theorie der Gesellschaftsspiele". In the list of references the two publications are listed in chronological order. Whenever the phrase: "the 1928 paper" occurs in the text it refers to [von Neumann, 1928].

1928, p. 295] and sent the manuscript to *Mathematische Annalen* in July 1927, it was not until May 1928, that he saw to it that his work was presented to the *Académie des Sciences* in Paris [von Neumann, 1928a]. If von Neumann had known about Borel's work all the time it does not make sense to wait more than a year to have his own ideas presented at the *Académie* in Paris, it would have been more natural to have it presented right away. Instead von Neumann had it prestented just a short time before the paper itself was published, which makes perfect sense assuming he learned about Borel's work for the first time during the proofreading stage. At that point he might have been afraid that Borel or someone else would be about to publish a similar result and then acted quickly by sending a note to the *Académie* to ensure priority. The other explanation given in the literature for why von Neumann suddenly developed a theory of games is more plausible I think. Here the apperance of von Neumann's game-theoretic work is linked with the social context of von Neumann's life during the years leading up to the publication of the paper. Von Neumann was at that time very much influenced by Hilbert and the Göttingen mathematical community. He was especially deeply involved in Hilbert's axiomatization programme.[10] In the papers [Mirowski, 1991, 1992] Mirowski argues convincingly that von Neumann's game theory was a result of his connection to Hilbert and the formalist programme.[11]

2.1. *What is a "Gesellschaftsspiel"?*

The two essential parts of von Neumann's 1928 paper are the mathematization of "Gesellschaftsspiele" or "games of strategy" and the proof of the theorem "Max Min = Min Max" for a game involving two players who play against each other and for which the players' gains add up to zero. That is the theorem now known as the minimax theorem for two-person zero-sum games. In the following I will explain how von Neumann mathematized games of strategy and how he proved the minimax theorem.[12]

Von Neumann began the paper by posing the question under consideration

> n Spieler, $S_1, S_2, \ldots, S_n$, spielen ein gegebenes Gesellschaftsspiel B. Wie muß einer dieser Spieler, S_m, spielen, um dabei ein möglichst günstiges Resultat zu erzielen?[13] [von Neumann, 1928, p. 295]

As von Neumann pointed out, the problem is well known from daily life but ambiguous because what will happen when there are more than one player involved? In that case the fate of each player depends on the rest of the players and they are all guided by the

[10] In 1925–1928 he published three papers on the axiomatization of set theory, one on Hilbert's proof theory, and seven papers on the foundation and axiomatization of quantum mechanics; see the bibliography of John von Neumann in his collected works [von Neumann, 1963, pp. 645–652].

[11] See also [Leonard, 1992, 1995].

[12] The paper was published in German. In 1959 an English translation of it was published from which the translations in the footnotes of the quotes have been taken.

[13] "n players $S_1, S_2, \ldots, S_n$ are playing a game of strategy, B. How must one of the participants, S_m, play in order to achieve a most advantageous result?" [von Neumann, 1928, (1959 p. 13)].

same selfish interests. Thus the first problem von Neumann faced was to clarify what precisely was to be understood by the term "Gesellschaftsspiel". As the following quote shows, von Neumann had a very broad understanding of the concept

> Es fallen unter diesen Begriff sehr viele, recht verschiedenartige Dinge: von der Roulette bis zum Schach, vom Bakkarat bis zum Bridge liegen ganz verschiedene Varianten des Sammelbegriffes 'Gesellschaftsspiel' vor. Und letzten Endes kann auch irgendein Ereignis, mit gegebenen äusseren Bedingungen und gegebenen Handelnden (den absolut freien Willen der letzteren vorausgesetzt), als Gesellschaftsspiel angesehen werden, wenn man seine Rückwirkungen auf die in ihm handelnden Personen betrachtet.[14] [von Neumann, 1928, p. 295]

Von Neumann's very broad interpretation of "Gesellschaftsspiele" points towards an extremely ambitious project. At a first glance it must have seemed very unlikely that one could succeed in building a mathematical model for this kind of situation. Anyway, even though von Neumann was 'only' able to contruct a solution concept and prove the existence of such a solution for a very limited subset of the overall game concept, he started out with the mathematization of the general case.

By collecting the common features in game situations von Neumann first derived what he called a qualitative description of the game concept. He argued as follows: A game is composed of a series of events of which each can have at most a finite number of outcomes. In some game situations it can happen that the outcome of some of the events depends only on chance. This means that the probabilities with which each of the outcomes will appear are known but none of the players have any influence on them. The outcome of all other events are subject to the individual player's free choices. For each of these events it is known which player determines the outcome, and what kind of information this player has regarding the outcome of earlier events. Finally there is a rule by which the gains and losses of each player can be calculated after the game, that is after the outcome of all events in the play are known. [von Neumann, 1928, p. 296].

In order to be able to work with this very broad concept of a game von Neumann reformulated the above description in a more precise form which then served as his definition of a game of strategy. His definition was build up around five points.

The first one specifies the number (z) of events depending on chance and the number (s) of events depending on the free will of the players. Von Neumann let

$$E_1, E_2, \ldots, E_z$$

denote the events depending on chance, and

$$F_1, F_2, \ldots, F_s$$

denote the events depending on the free will of the players.

The second is the specification of the number M_μ $(\mu = 1, 2, \ldots, z)$ of possible outcomes of each single event of chance E_μ, and the number N_ν $(\nu = 1, 2, \ldots, s)$ of

---

[14] "A great many different things come under this heading, anything from roulette to chess, from baccarat to bridge. And after all, any event – given the external conditions and the participants in the situation (provided the latter are acting of their own free will) – may be regarded as a game of strategy if one looks at the effect it has on the participants." [von Neumann, 1928 (1959 p. 13)].

possible outcomes of each single event of free will F_ν. Von Neumann referred to a result of an event by its number, i. e. $1, 2, \ldots, M_\mu$ or $1, 2, \ldots, N_\nu$.

The third thing one needs to know in von Neumann's game model is the probabilities $\alpha_\mu^{(1)}, \alpha_\mu^{(2)}, \ldots, \alpha_\mu^{(M_\mu)}$ with which the outcomes $1, 2, \ldots, M_\mu$ of an event of chance E_μ will occur, thus

$$\alpha_\mu^{(1)} \geq 0, \alpha_\mu^{(2)} \geq 0, \ldots, \alpha_\mu^{(M_\mu)} \geq 0,$$

and

$$\alpha_\mu^{(1)} + \alpha_\mu^{(2)} + \cdots + \alpha_\mu^{(M_\mu)} = 1.$$

For every event of free will F_ν, one also needs to specify which player S_m determines the outcome of this event and in addition one also needs to know what events have occurred up to this moment, that is the corresponding numbers for all earlier events, both those of chance and those of free will that the player in charge has information about when he or she makes up his or her mind.

Finally one needs to specify n real valued functions $f_1, \ f_2, \ldots, \ f_n$ of $z+s$ variables, where the first z variables can take the values

$$1, 2, \ldots, M_1; \quad 1, 2, \ldots, M_2; \quad \ldots; \quad 1, 2, \ldots, M_z;$$

and the last s variables can take the values

$$1, 2, \ldots, N_1; \quad 1, 2, \ldots, N_2; \quad \ldots; \quad 1, 2, \ldots, N_s.$$

These functions determine the gain of the players and must add up to zero

$$f_1 + f_2 + \cdots + f_n \equiv 0.$$

Suppose the results of the z events of chance and the s events of free will in a game turn out to be

$$x_1, x_2, \ldots, x_z, \quad y_1, y_2, \ldots, y_s,$$

respectively, where

$$x_\mu \in \{1, 2, \ldots, M_\mu\}, \quad y_\nu \in \{1, 2, \ldots, N_\nu\},$$

$$\mu = 1, 2, \ldots, z, \quad \nu = 1, 2, \ldots, s,$$

the players $S_1, S_2, \ldots, S_n$ then 'gain' the amounts

$$f_1(x_1, \ldots, x_z, y_1, \ldots, y_s), \ f_2(x_1, \ldots, x_z, y_1, \ldots, y_s), \ \ldots,$$

$$f_n(x_1, \ldots, x_z, y_1, \ldots, y_s)$$

[von Neumann, 1928, p. 296–297].

In this way von Neumann defined a game of strategy. But as he himself pointed out, the notion of a player S_m, trying to achieve a result as advantageous as possible is kind of obscure. It is clear that the most advangeous result for S_m has to be defined as the largest possible value of f_m, but f_m depends of $z + s$ variables of which only a part is controlled by S_m, and this is exactly the heart of the problem:

> Es soll versucht werden, die Rückwirkungen der Spieler aufeinander zu untersuchen, die Konsequenzen des (für alles soziale Geschehen so charakteristischen!) Umstandes, daß jeder Spieler auf die Resultate aller anderen einen Einfluß hat und dabei nur am eigenen interessiert ist.[15] [von Neumann, 1928, p. 298]

The next step in von Neumann's building of the theory was to simplify the game concept as much as possible without loosing anything in generality. The key trick was to introduce the concept of strategy. By a strategy of a player S_m von Neumann understood a complete plan for how this player is going to act in a precisely defined situation. It can be thought of as a collection of decisions for any conceivable future situation of the game [von Neumann, 1928, p. 18]. All the information about the other players' decisions and the outcome of the events of chance that a player has access to is inherent in the concept of strategy. The consequence of this is, that each player chooses his or her strategy knowing neither the strategies chosen by the others nor the results of the events of chance. As noticed by von Neumann each player S_m has only a finite number, Σ_m, of strategies $(S_1^{(m)}, S_2^{(m)}, \ldots, S_{\Sigma_m}^{(m)})$ to choose from. That is, by introducing the concept of strategy von Neumann reduced the number of free choices to the number of players, such that the choice number ν is determined by the free will of player S_ν.

Another advantage of the concept of strategy is, that von Neumann could eliminate the events of chance altogether. First he reduced the number of events of chance to one. Because, since a player has to choose the strategy without knowing beforehand the outcome of the events of chance, these events need no longer be treated as separate events. It is then possible to combine all z events of chance into one single event of chance H, the outcome of which will be a collection of numbers

$$x_1,\ x_2, \ldots, x_z \quad (x_\mu \in \{1, 2, \ldots, M_\mu\})$$

with their respective probabilities

$$\alpha_1^{(x_1)} \alpha_2^{(x_2)} \ldots \alpha_z^{(x_z)}.$$

There are $\hat{M} = M_1 M_2 \cdot \cdots \cdot M_z$ possible collections of these numbers.[16] Von Neumann associated each collection with a number

$$1, 2, \ldots, \hat{M} \quad (\hat{M} = M_1 M_2 \cdot \cdots \cdot M_z),$$

and he let

$$\beta_1, \beta_2, \ldots, \beta_{\hat{M}}$$

denote the corresponding probabilities [von Neumann, 1928, p. 300].

[15] "We shall try to investigate the effects which the players have on each other, the consequences of the fact (so typical of all social happenings!) that each player influences the results of all other players, even though he is only interested in his own" [von Neumann, 1928, (1959 p. 17)].

[16] Von Neumann used the symbol M instead of $\hat{M}$. Later, in Sect. 2.3, M means something else, and to avoid confusion I have here used the symbol $\hat{M}$.

By doing so von Neumann had boiled it all down to the following: If the players $S_1, S_2, \ldots, S_n$ have chosen the strategies

$$S^{(1)}_{u_1}, S^{(2)}_{u_2}, \ldots, S^{(n)}_{u_n},$$

where

$$u_m \in \{1, 2, \ldots, \Sigma_m\}, \quad m = 1, 2, \ldots, n,$$

and if the outcome of the event of chance H, is the number ν ($\nu \in \{1, 2, \ldots, \hat{M}\}$), then the results for the players $S_1, S_2, \ldots, S_n$ are

$$f_1(\nu, u_1, u_2, \ldots, u_n), \; f_2(\nu, u_1, u_2, \ldots, u_n), \ldots, f_n(\nu, u_1, u_2, \ldots, u_n)$$

respectively. Now if only the choices $u_1, u_2, \ldots, u_n$, and not the result ν, of the event of chance is known, then the expected value of f_m would be

$$g_m(u_1, \ldots, u_n) = \sum \beta_\nu f_m(\nu, u_1, \ldots, u_n), \quad (m = 1, 2, \ldots, n),$$

($f_1 + \cdots + f_n \equiv 0$ implies $g_1 + \cdots + g_n \equiv 0$). Von Neumann then argued that according to the theory of probability it is fully acceptable to ignore the events of chance and instead work with the expected values $g_1, \ldots, g_n$. That is, by substituting the *exact* results (f_m) for the individual players by the *expected* values he elimated H altogether.

These simplifications left von Neumann with the following formulation of a game of strategy: Each of the players $S_1, S_2, \ldots, S_n$ chooses a number without any information about the choice of the others. S_m chooses among the numbers $1, 2, \ldots, \Sigma_m$.[17] After the choices $x_1, x_2, \ldots, x_n$ ($x_m \in \{1, 2, \ldots, \Sigma_m\}$) have been made the players will receive the amount

$$g_1(x_1, \ldots, x_n), \; g_2(x_1, \ldots, x_n), \ldots, \; g_n(x_1, \ldots, x_n),$$

respectively, where $g_1 + \cdots + g_n = 0$ holds [von Neumann, 1928, p. 301–302].

2.2. The case $n = 2$

In 1928 von Neumann was not able to prove anything about the existence of optimal strategies for the general case. In stead he analyzed the simplest case, namely a game of strategy with only two players S_1 and S_2. The situation is then, that player S_1 chooses a number $x \in \{1, 2, \ldots, \Sigma_1\}$, and player S_2 chooses a number $y \in \{1, 2, \ldots, \Sigma_2\}$ each without knowing what the other player has chosen, and they then receive the amounts $g(x, y), -g(x, y)$ respectively. Von Neumann then gave the following description of the tension in the two-person game:

> Es ist leicht, sich ein Bild von den Tendenzen zu machen, die in einem solchen 2-Personen-Spiele miteinander kämpfen: Es wird von zwei Seiten am Werte von $g(x, y)$ hin und her gezerrt, nämlich durch S_1, der ihn möglichst gross, und durch S_2, der ihn möglichst

[17] Note that here the strategies are associated with their number, such that if player S_m choose the number i it means that he or she will play according to strategy $S_i^{(m)}$.

klein machen will. S_1, gebietet über die Variable x, und S_2 über die Variable y. Was wird geschehen?[18] [von Neumann, 1928, p. 302]

The core question is 'What will happen?' Von Neumann's analysis of the situation ran as follows: If S_1 chose the number x_0 ($x_0 \in \{1, 2, \ldots, \Sigma_1\}$), that is the strategy x_0, his result $g(x_0, y)$ would then also depend on the choice of S_2; but no matter which choice (y) S_2 comes up with, the following inequality holds:

$$g(x_0, y) \geq \min_y g(x_0, y).$$

Now if we suppose (against the rules of the game) that S_2 knew x_0, S_2 would according to the assumptions in the model choose $y = y_0$ such that

$$g(x_0, y_0) = \min_y g(x_0, y).$$

Facing this situation the best thing for S_1 would be to choose x_0 such that

$$\min_y g(x_0, y) = \max_x \min_y g(x, y).$$

The conclusion of von Neumann is then that S_1 can make

$$g(x_0, y) \geq \max_x \min_y g(x, y),$$

independently of the choise of S_2. The same argument holds for S_2, which can make

$$g(x, y_0) \leq \min_y \max_x g(x, y),$$

no matter what strategy x, S_1 chooses.

From this von Neumann concluded that if a pair of strategies x_0, y_0 can be found for which

$$g(x_0, y_0) = \max_x \min_y g(x, y) = \min_y \max_x g(x, y) = \tilde{M},$$

then that would necessary be the choices for S_1 and S_2 respectively, and $\tilde{M}$ would be the result of the game[19] [von Neumann, 1928, pp. 302–303]. Thus, such a pair of strategies x_0, y_0, if they exist, would constitute a solution concept for two-person games. Unfortunately the existence of such a pair of strategies is not automatically guaranteed.

The trick used by von Neumann to overcome this difficulty was to introduce what is now known as mixed strategies. Instead of choosing an x or a y, the players specify the probabilities with which they will choose the different strategies. That is, the player S_1 chooses Σ_1 probabilities

$$\xi_1, \xi_2, \ldots, \xi_{\Sigma_1} \quad \left(\xi_1 \geq 0, \xi_2 \geq 0, \ldots, \xi_{\Sigma_1} \geq 0, \quad \sum \xi_i = 1\right),$$

---

[18] "It is easy to picture the forces struggling with each other in such a two-person game. The value of $g(x, y)$ is being tugged at from two sides, by S_1 who wants to maximize it, and by S_2 who wants to minimize it. S_1 controls the variable x, S_2 the variable y. What will happen?" [von Neumann, 1928, (1959 p. 21)].

[19] Von Neumann used the symbol M instead of $\tilde{M}$. Later, in Sect. 2.3, M means something else; to avoid confusion I have here used the symbol $\tilde{M}$, cf. note 16.

and from an urn containing the numbers $1, 2, \ldots, \Sigma_1$ with the above specified probabilities, he or she draws a number and chooses that number. Analogously, S_2 specifies Σ_2 probabilities

$$\eta_1, \eta_2, \ldots, \eta_{\Sigma_2} \quad \left(\eta_1 \geq 0, \eta_2 \geq 0, \ldots, \eta_{\Sigma_2} \geq 0, \quad \sum \eta_j = 1\right).$$

Von Neumann put

$$(\xi_1, \xi_2, \ldots, \xi_{\Sigma_1}) = \xi, \text{ and } (\eta_1, \eta_2, \ldots, \eta_{\Sigma_2}) = \eta.$$

If S_1 chooses ξ, and S_2 chooses η, the expected value of the amount S_1 receives is

$$h(\xi, \eta) = \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g(p, q) \xi_p \eta_q, \tag{1}$$

while the expected value for S_2 is $-h(\xi, \eta)$ [von Neumann, 1928, p. 304].

As before von Neumann argued that S_1 is in a position to obtain the minimal expected value $\max_\xi \min_\eta h(\xi, \eta)$ no matter what S_2 chooses to do. S_2 can keep the expected value of S_1 from exceeding the maximal value $\min_\eta \max_\xi h(\xi, \eta)$. By considering the mixed strategies instead of pure strategies the expected values of the players is expressed by the bilinear form h, and for those von Neumann was able to show that there always exist mixed strategies ξ_0, η_0 such that

$$\max_\xi \min_\eta h(\xi, \eta) = \min_\eta \max_\xi h(\xi, \eta) = h(\xi_0, \eta_0).$$

This result is the famous minimax theorem of von Neumann and it establishes that for two-person games of this kind there always exist optimal (mixed-) strategies. This is called the minimax solution concept of two-person zero-sum games. It has been criticized for being too defensive a solution concept, indeed it is a solution telling you what is the best you can do in the worst possible case.

2.3. *Von Neumann's proof of the minimax theorem*

Actually von Neumann proved a generalized version of the minimax theorem. He considered a broader class of functions than the bilinear forms h. For continuous functions f of two variables $\xi \in \mathbf{R}^M$, $\eta \in \mathbf{R}^N$, $\xi \geq 0$, $\eta \geq 0$, $\xi_1 + \cdots + \xi_M \leq 1$, $\eta_1 + \cdots + \eta_N \leq 1$ satisfying the condition:

> (**K**.) Wenn $f(\xi', \eta) \geq A$, $f(\xi'', \eta) \geq A$ ist, so ist auch für jedes $0 \leq \nu \leq 1$, $\xi = \nu\xi' + (1 - \nu)\xi''$ (d.h. $\xi_p = \nu\xi'_p + (1 - \nu)\xi''_p$, $p = 1, 2, \ldots, M)$ $f(\xi, \eta) \geq A$. Wenn $f(\xi, \eta') \leq A$, $f(\xi, \eta'') \leq A$ ist, so ist auch für jedes $0 \leq \nu \leq 1$, $\eta = \nu\eta' + (1 - \nu)\eta''$ (d.h. $\eta_q = \nu\eta'_q + (1 - \nu)\eta''_q$, $q = 1, 2, \ldots, N)$ $f(\xi, \eta) \leq A$. [von Neumann, 1928, p. 307]

he formulated the theorem in the following way:

> ... Für diese Funktionen $f(\xi, \eta)$ werden wir beweisen:
>
> $$\max_\xi \min_\eta f(\xi, \eta) = \min_\eta \max_\xi f(\xi, \eta),$$
>
> wobei $\max_\xi$ über $\xi_1 \geq 0, \ldots, \xi_M \geq 0, \xi_1 + \cdots + \xi_M \leq 1$, und $\min_\eta$ über $\eta_1 \geq 0, \ldots, \eta_N \geq 0$, $\eta_1 + \cdots + \eta_N \leq 1$ zu erstrecken ist.[20] [von Neumann, 1928, p. 307]

Today a function with the property (**K**) is called quasiconcave in ξ and quasiconvex in η.

Since the function $h(\xi, \eta)$, that determines the expected values for the two players is bilinear, it is also continuous and has the property (**K**), so a proof of this theorem will also prove the existence of optimal strategies for a two-person zero-sum game.

Von Neumann began by rewriting

$$\max_\xi \min_\eta f(\xi, \eta) = \min_\eta \max_\xi f(\xi, \eta)$$

in the form

$$\begin{aligned} &\max_{\substack{\xi_1 \geq 0 \\ \xi_1 \leq 1}} \max_{\substack{\xi_2 \geq 0 \\ \xi_1+\xi_2 \leq 1}} \ldots \max_{\substack{\xi_M \geq 0 \\ \xi_1+\cdots+\xi_M \leq 1}} \min_{\substack{\eta_1 \geq 0 \\ \eta_1 \leq 1}} \min_{\substack{\eta_2 \geq 0 \\ \eta_1+\eta_2 \leq 1}} \ldots \min_{\substack{\eta_N \geq 0 \\ \eta_1+\cdots+\eta_N \leq 1}} f(\xi, \eta) \\ = &\min_{\substack{\eta_1 \geq 0 \\ \eta_1 \leq 1}} \min_{\substack{\eta_2 \geq 0 \\ \eta_1+\eta_2 \leq 1}} \ldots \min_{\substack{\eta_N \geq 0 \\ \eta_1+\cdots+\eta_N \leq 1}} \max_{\substack{\xi_1 \geq 0 \\ \xi_1 \leq 1}} \max_{\substack{\xi_2 \geq 0 \\ \xi_1+\xi_2 \leq 1}} \ldots \max_{\substack{\xi_M \geq 0 \\ \xi_1+\cdots+\xi_M \leq 1}} f(\xi, \eta). \end{aligned} \tag{2}$$

By putting

$$M^{\xi_r} f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s) = \max_{\xi_1+\cdots+\xi_r \leq 1} {}_{\xi_r} f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s),$$

$$M^{\eta_s} f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s) = \min_{\eta_1+\cdots+\eta_s \leq 1} {}_{\eta_s} f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s),$$

he eliminated f's dependency on ξ_r and η_s respectively. Thus von Neumann wrote the identity under consideration as

$$M^{\xi_1} M^{\xi_2} \ldots M^{\xi_p} M^{\eta_1} M^{\eta_2} \ldots M^{\eta_q} f = M^{\eta_1} M^{\eta_2} \ldots M^{\eta_q} M^{\xi_1} M^{\xi_2} \ldots M^{\xi_p} f.$$

With $p = M$ and $q = N$ this is equivalent to von Neumann's formulation above (2) of the minimax theorem, where he considered $\xi \in \mathbf{R}^M$ and $\eta \in \mathbf{R}^N$.

With these reformulations as a tool von Neumann reduced the proof to the proof of the following two lemmas:

---

[20] "(**K.**) If $f(\xi', \eta) \geq A$, $f(\xi'', \eta) \geq A$, then $f(\xi, \eta) \geq A$ for every $0 \leq \nu \leq 1$, $\xi = \nu\xi' + (1-\nu)\xi''$ (i.e., $\xi_p = \nu\xi'_p + (1-\nu)\xi''_p$, $p = 1, 2, \ldots, M$). If $f(\xi, \eta') \leq A$, $f(\xi, \eta'') \leq A$, then $f(\xi, \eta) \leq A$ for every $0 \leq \nu \leq 1$, $\eta = \nu\eta' + (1-\nu)\eta''$ (i.e., $\eta_q = \nu\eta'_q + (1-\nu)\eta''_q$, $q = 1, 2, \ldots, N$).

... For these functions $f(\xi, \eta)$ we are going to prove that

$$\max_\xi \min_\eta f(\xi, \eta) = \min_\eta \max_\xi f(\xi, \eta),$$

where $\max_\xi$ is taken over the range $\xi_1 \geq 0, \ldots, \xi_M \geq 0$, $\xi_1 + \cdots + \xi_M \leq 1$ and $\min_\eta$ is taken over the range $\eta_1 \geq 0, \ldots, \eta_N \geq 0$, $\eta_1 + \cdots + \eta_N \leq 1$." [von Neumann, 1928, (1959 p. 26–27)].

α) If $f = f(\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_s)$ is continuous and has the property (**K**) then $M^{\xi_r} f$ and $M^{\eta_s} f$ are continuous and satisfy (**K**).

β) If $f = f(\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_s)$ is continuous and satisfy (**K**) then

$$M^{\xi_r} M^{\eta_s} f = M^{\eta_s} M^{\xi_r} f.$$

It is easy to see that the minimax theorem can be derived directly from (α) and (β).

Von Neumann's proof of (α) is straightforward. The continuity of $M^{\xi_r} f$ and $M^{\eta_s} f$ is a direct consequence of the continuity of f. To prove that $M^{\xi_r} f$ and $M^{\eta_s} f$ has the property (**K**) von Neumann used first that a continuous function on a closed and boundet set has a maximum and a minimum and secondly that f itself has the property (**K**).[21]

The central part of the proof is (β) which is much more complicated to prove. In what follows I will go through the proof step by step following von Neumann closely. Then I will comment on the proof and discuss it in relation to the description given by Kuhn and Tucker.

The proof for (β)

Von Neumann is going to prove that $M^{\xi_r} M^{\eta_s} f = M^{\eta_s} M^{\xi_r} f$ for all $\xi_1, \dots, \xi_{r-1}$, $\eta_1, \dots, \eta_{s-1}$. He began by considering f for some fixed $\xi_1, \dots, \xi_{r-1}$, $\eta_1, \dots, \eta_{s-1}$. Then f is a function of ξ_r and η_s alone and f is obviously still continuous and posses the property (**K**). Writing ξ and η instead of ξ_r and η_s respectively what von Neumann is going to show is that

$$\max_{0 \le \xi \le a} \min_{0 \le \eta \le b} f(\xi, \eta) = \min_{0 \le \eta \le b} \max_{0 \le \xi \le a} f(\xi, \eta),$$

where $a = 1 - \xi_1 - \cdots - \xi_{r-1}$ and $b = 1 - \eta_1 - \cdots - \eta_{s-1}$. As pointed out by von Neumann this can also be formulated in an other way:

> Es gibt einen "Sattelpunkt" ξ_0, η_0 ($0 \le \xi_0 \le a$, $0 \le \eta_0 \le b$), d.h. $f(\xi_0, \eta)$ nimmt in $0 \le \eta \le b$ sein Minimum für $\eta = \eta_0$ an, und $f(\xi, \eta_0)$ nimmt in $0 \le \xi \le a$ sein Maximum für $\xi = \xi_0$ an.[22] [von Neumann, 1928, p. 309]

Now it is always true that

$$\max_\xi \min_\eta f(\xi, \eta) \le \min_\eta \max_\xi f(\xi, \eta).$$

On the other hand, if there exists a saddle point (ξ_0, η_0), then

$$\max_\xi \min_\eta f(\xi, \eta) \ge \min_\eta f(\xi_0, \eta) = f(\xi_0, \eta_0),$$

$$\min_\eta \max_\xi f(\xi, \eta) \le \max_\xi f(\xi, \eta_0) = f(\xi_0, \eta_0)$$

which gives the other inequality, hence

$$\max_\xi \min_\eta f(\xi, \eta) = \min_\eta \max_\xi f(\xi, \eta) = f(\xi_0, \eta_0).$$

With this, what needs to be proven is the existence of such a saddlepoint.

---

[21] For the detailed proof see [von Neumann, 1928, p. 308–309].

[22] "There exists a "saddle point" ξ_0, η_0 ($0 \le \xi_0 \le a$, $0 \le \eta_0 \le b$), i.e., $f(\xi_0, \eta)$ assumes its minimum for $\eta = \eta_0$ in $0 \le \eta \le b$ and $f(\xi, \eta_0)$ assumes its maximum for $\xi = \xi_0$ in $0 \le \xi \le a$." [von Neumann, 1928, (1959 p. 30)].

The existence of a saddle point

For every fixed ξ von Neumann considered the set of η, $0 \leq \eta \leq b$, for which $f(\xi, \eta)$ assumes its minimum value. Since f is continuous the set will be closed and it will also be convex because f satisfies the condition **(K)**; this means the set is a subinterval of $[0, b]$. Von Neumann let $[K'(\xi), K''(\xi)]$ denote this subinterval. Thus for fixed ξ:

$$\{\eta' \in [0, b] | \min_\eta f(\xi, \eta) = f(\xi, \eta')\} = [K'(\xi), K''(\xi)] \subseteq [0, b].$$

Similarly for fixed η, the set of ξ, $0 \leq \xi \leq a$, for which $f(\xi, \eta)$ assumes its maximum, is a closed subinterval of $[0, a]$. Von Neumann denoted this subinterval by $[L'(\eta), L''(\eta)]$.

That is, for every $\xi \in [0, a]$ there is an interval $[K'(\xi), K''(\xi)] \subseteq [0, b]$, such that every single η belonging to this interval is a point of minimum for the function $f(\xi, *)$. Similarly for every $\eta \in [0, b]$ there is an interval $[L'(\eta), L''(\eta)] \subseteq [0, a]$, such that every ξ in this interval is a point of maximum for the function $f(*, \eta)$.

Von Neumann then showed that – due to the continuity of f – K', L' and K'', L'' are lower and upper semi-continuous functions respectively. [von Neumann, 1928, p. 310, note 10].

For a fixed ξ^* von Neumann studied the following set which I have named $D(\xi^*)$:

$$D(\xi^*) = \{\xi^{**} | \exists \eta^* : \min_\eta f(\xi^*, \eta) = f(\xi^*, \eta^*) \text{ and } \max_\xi f(\xi, \eta^*) = f(\xi^{**}, \eta^*)\},$$

that is,

$$D(\xi^*) = \cup[L'(\eta^*), L''(\eta^*)] \text{ over } \eta^* \in [K'(\xi^*), K''(\xi^*)].$$

Within the interval $K'(\xi^*) \leq \eta^* \leq K''(\xi^*)$, the lower semi-continuous function L' will assume its minimum value and the upper semi-continuous function L'' will assume its maximum value. Hence the set $D(\xi^*)$ will contain a minimal as well as a maximal element. Furthermore von Neumann argued by means of the following indirect proof that $D(\xi^*)$ also contains all ξ' between the minimal and the maximal element: In contradiction to what he wanted to demonstrate von Neumann assumed the existence of an element ξ' situated in between the minimal and the maximal element but not contained in $D(\xi^*)$. Then every interval $[L'(\eta^*), L''(\eta^*)]$ would lie either entirely to the left or entirely to the right of ξ'. Since ξ' is between the minimum and the maximum element of $D(\xi^*)$, both kind of intervals will exist. η^* runs over an interval, which implies that both kinds of η^*'s, that is, those η^*'s corresponding to the intervals $[L'(\eta^*), L''(\eta^*)]$ entirely to the left of ξ', and those η^*'s corresponding to the intervals $[L'(\eta^*), L''(\eta^*)]$ entirely to the right of ξ', have a common limit-point η'. This means that both $L'(\eta^*) \leq \xi'$ and $L''(\eta^*) \geq \xi'$ will occur arbitrary close to η', which because of the lower and upper semi-continuity of L' and L'' respectively, implies that $L'(\eta') \leq \xi'$ and $L''(\eta') \geq \xi'$, that is ξ' does indeed belong to one of the intervals, namely $[L'(\eta'), L''(\eta')]$ [von Neumann, 1928, p. 310].

The above result implies that $D(\xi^*)$ is a closed subinterval of $[0, a]$, which von Neumann denoted $[H'(\xi^*), H''(\xi^*)]$. To finish the demonstration he showed the existence of an element $\xi^* \in [0, a]$, which is also a ξ^{**}, that is, an element ξ^*, for which $H'(\xi^*) \leq \xi^* \leq H''(\xi^*)$. The proof for this is similar to the proof above for the claim that $D(\xi^*)$ is a closed subinterval, due to the fact that H' and H'' are lower and upper semi-continuous functions respectively. As before, if one assumes that there can exist no such ξ^* that would imply that all the intervals $[H'(\xi^*), H''(\xi^*)]$ will lie entirely to the left or entirely to the right of ξ^*. Again both kinds of ξ^*'s will have a common limit point ξ' which will belong to the interval $[H'(\xi'), H''(\xi')]$.

With this von Neumann has demonstrated the existence of an element $\xi^* \in [0, a]$, which also satisfies $\xi^* \in D(\xi^*)$. Since this means that there exists an element η^*, such that $\min_\eta f(\xi^*, \eta) = f(\xi^*, \eta^*)$, and at the same time $\max_\xi f(\xi, \eta^*) = f(\xi^*, \eta^*)$, the point ξ^*, η^* is a saddle point for the function f, which finished von Neumann's proof of the minimax theorem.

2.4. Von Neumann's 1928 proof in relation to fixed point theorems and systems of inequalities

The 1928 proof of von Neumann is indeed a "tour de force" [Heims, 1980, p. 91]. Summarizing the other remarks in the literature that I cited in the introduction it has been said about von Neumann's 1928-proof that he "demonstrates the close connection with fixed point theorems and especially Brouwer's theorem" [Ingrao and Israel, 1990, p. 211], and that the proof concerns the existence of a solution to a system of equalities and inequalities.[23] These statements do not seem very obvious to me. Von Neumann talked at no point about fixed points and he did not formulate a system of equations and inequalities to be solved.

Yet, in Kuhn and Tucker's paper about von Neumann's work on game theory they wrote:

> The analytic proofs of the Minimax Theorem given by von Neumann were of two essentially different types. Proofs of the first type (see $[A]$ and $[B]$) are based explicitly on extensions of the Brouwer's fixed point theorem; [Kuhn and Tucker, 1958, p. 112]

$[A]$ refers to von Neumann's 1928 paper while $[B]$ refers to a paper by von Neumann published in 1937 which will be treated in the next section. From von Neumann's 1928 proof one can "extract" a proof for an extension of Brouwer's fixed point theorem. In doing so the question about existence of a saddle point for the function $f(\xi, \eta)$ becomes a question about the existence of a fixed point for a 'point to set' map. The connection can be derived in the following way:

In the end von Neumann showed the existence of an element ξ^* satisfying $\xi^* \in [H'(\xi^*), H''(\xi^*)]$. If one puts $F(\xi^*) = [H'(\xi^*), H''(\xi^*)]$, F can be interpreted as a map which associates to each element ξ in $[0, a]$ a set, namely the subinterval $F(\xi)$ of $[0, a]$. An element ξ^* which is mapped onto an interval $F(\xi^*)$ to which the element itself belongs, is a kind of a fixed point for a 'point to set' map. With this interpretation the existence of a saddle point and the existence of a fixed point for the mapping F is one and the same thing.

Von Neumann did not make this interpretation in the 1928 paper and for reasons to be discussed in the next section I am not convinced that von Neumann in 1928 was aware of this connection to fixed points.

---

[23] See [Ingrao and Israel, 1990, p. 211], [Heims, 1980, p. 91].

As far as the connection to systems of linear inequalities and equations is concerned, von Neumann in his proof did not draw any connections at all. But as Kuhn and Tucker show in their essay on von Neumann's work it is possible to derive such a connection. It can be done in the following way: In an analysis of the consequences of the minimax theorem for the choice of strategies von Neumann considered the set A of all ξ for which $\min_\eta h(\xi,\eta)$ assumes its maximum value $\tilde{M}$, and the set B of all η for which $\max_\xi h(\xi,\eta)$ assumes its minimum value $\tilde{M}$. That is,

$$A = \{\xi \in \mathbf{R}^{\Sigma_1} : \min_\eta h(\xi,\eta) \text{ assumes its maximum value } \tilde{M}\}$$

$$B = \{\eta \in \mathbf{R}^{\Sigma_2} : \max_\xi h(\xi,\eta) \text{ assumes its minimum value } \tilde{M}\}.$$

As pointed out by von Neumann it is obvious that

1. if ξ belongs to A then $h(\xi,\eta) \geq \tilde{M}$ applies for all η (because $h(\xi,\eta) \geq \min_\eta h(\xi,\eta) = \tilde{M}$, since ξ belongs to A),
2. similar, if η belongs to B then $h(\xi,\eta) \leq \tilde{M}$ for all ξ,
3. if ξ does not belong to A there exists an element η for which $h(\xi,\eta) < \tilde{M}$,
4. if η does not belong to B there exists an element ξ for which $h(\xi,\eta) > \tilde{M}$,
5. if ξ belongs to A and η belongs to B then $h(\xi,\eta) = \tilde{M}$.

Hence, von Neumann argued, it is obvious that S_1 should choose a strategy ξ that belongs to A and S_2 should choose a strategy η which belongs to B. For every such choice the game will have the value $\tilde{M}$ for S_1 and the value $-\tilde{M}$ for S_2 [von Neumann, 1928, p. 305].

In the 1928 paper von Neumann did not discuss this further, but one can interpret this as the finding of elements ξ^*, η^*, such that the inequalities

$$\xi^* \geq 0, \quad \eta^* \geq 0, \quad \max_\xi h(\xi,\eta^*) \leq \tilde{M}, \quad \min_\eta h(\xi^*,\eta) \geq \tilde{M} \tag{3}$$

and the equalities

$$\xi_1{}^* + \cdots + \xi_{\Sigma_1}{}^* = 1,\ \eta_1{}^* + \cdots + \eta_{\Sigma_2}{}^* = 1 \tag{4}$$

are all satisfied.

Kuhn and Tucker in their essay derived the following connection between solutions $(\xi^*,\eta^*,\tilde{M})$ to the 'minimax problem' and a system of linear inequalities and equations. They wrote the bilinear form (1) (introduced in section 2.2) as $h(\xi,\eta) = \xi A\eta$, where A is a matrix with elements $g(p,q)$. Hence a solution (ξ^*,η^*) to the linear inequalities and equations

$$\xi^* \geq 0, \quad \eta^* \geq 0, \quad A\eta^* \leq \tilde{M}, \quad \xi^* A \geq \tilde{M}$$

$$\xi_1{}^* + \cdots + \xi_{\Sigma_1}{}^* = 1,\ \eta_1{}^* + \cdots + \eta_{\Sigma_2}{}^* = 1$$

will then also be a solution to the system (3) and (4) [Kuhn and Tucker, 1958, p. 111]. But this algebraical interpretation of optimal strategies as constituting a solution to a system of linear equalities and inequalities was not explicitly formulated by von Neumann in

1928. As we shall see in the next sections this insight came later. As a matter of fact von Neumann's 1928 proof is more general covering also nonlinear functions. To be fair it needs to be said that Kuhn and Tucker did not claim that von Neumann actually made this algebraic characterisation in 1928 but the other statements in the literature cited above leaves the impression that von Neumann in 1928 was working within a framework of linear inequality theory. As we shall see in the next sections and as von Neumann also himself later remarked about an announcement made by the French mathematician Fréchet, this connection to the theory of convexity and linear inequality theory was only recognized later on.

3. The connection to fixed point theorems and economy: von Neumann's 1937 paper

After the 1928 paper sixteen years passed before von Neumann published on game theory again. Yet the minimax theorem reappeared as early as 1932, but in another disguise. It happened in a mathematical-economic model that von Neumann developed in the early thirties. The first mention of the work is a talk von Neumann gave on the model at the mathematics seminar at Princeton. The paper was published five years later (at the request of Karl Menger) under the title "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes" [von Neumann, 1937].

The model of von Neumann is a linear production model in which he did not distinguish between goods consumed and goods produced in the process of production. He analyzed a situation where there are n goods $G_1, \ldots, G_n$ which can be produced by m processes $P_1, \ldots, P_m$; the numbers $y_1, \ldots, y_n$ denote the prices of the goods while $x_1, \ldots, x_m$ are the intensities with which the processes are being used. Finally a_{ij} and b_{ij} denote the number of units of the good G_j consumed and produced respectively by the process P_i.

Von Neumann was interested in situations where the whole economy expands without change of structure, i.e. where the ratios of the intensities $x_1 : \ldots : x_m$ remain unchanged, although $x_1, \ldots, x_m$ themselves may change [von Neumann, 1937, p. 30]. In such a case the intensities are multiplied by a common factor α per unit of time, the so-called coefficient of expansion. The unknowns are the intensities $x_1, \ldots, x_m$, the coefficient of expansion α, the prices $y_1, \ldots, y_n$ of the goods, and the interest factor $\beta = 1 + \frac{z}{100}$, where z is the rate of interest in % per unit of time [von Neumann, 1937, p. 30].

The analysis of von Neumann resulted in the following system of inequalities which was to be solved:

$$x_i \geq 0, \tag{5}$$

$$y_j \geq 0, \tag{6}$$

$$\sum_{i=1}^{m} x_i > 0, \tag{7}$$

$$\sum_{j=1}^{n} y_j > 0, \tag{8}$$

$$\alpha \sum_{i=1}^{m} a_{ij} x_i \leq \sum_{i=1}^{m} b_{ij} x_i, \quad \text{for all } j \tag{9}$$

where $y_j = 0$ if strict inequality '$<$' holds.

$$\beta \sum_{j=1}^{n} a_{ij} y_j \geq \sum_{j=1}^{n} b_{ij} y_j, \quad \text{for all } i \tag{10}$$

where $x_i = 0$ if strict inequality '$>$' holds.

The inequality (9) means that it is impossible to consume more of the good G_j than the amount produced in the total process. If more is produced than is consumed G_j becomes a free good with zero price $y_j = 0$. The inequality (10) appears because there is no profit in the model so a possible gain would be reinvested. (10) means that in equilibrium there cannot be a profit on any process P_i. If there is a loss, i.e. if '$>$' holds the process P_i will not be used and $x_i = 0$ [Von Neumann, 1937, p. 75–76].

3.1. *The solution of the system of inequalities*

In order to find necessary and sufficient conditions for the existence of a solution to such a system of linear inequalities von Neumann first transformed the problem of solutions into a saddle point problem. For this purpose he introduced the function

$$\phi(X, Y) = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} x_i y_j}{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j}$$

where $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_n)$ are variables satisfying (5), (7) and (6), (8) respectively. That is, he was looking at the ratio between the total income and the total costs.

Von Neumann then argued that the question of the existence of a solution to the system of inequalities (5)–(10) becomes the question of the existence of a saddle point for the function ϕ. Hence, he could formulate the question of the existence of a solution to the system (5)–(10) as follows:

(*) Consider (X, Y) in the domain bounded by (5)–(8). To find a saddle point $X = X_0$, $Y = Y_0$ for ϕ. (See [von Neumann, 1937, p. 78].)

Thus, just like in the 1928 paper on games, the key problem is to prove the existence of a saddle point for a certain function. Instead of proving the existence of a saddle point right away as he did in the 1928 paper he instead proved a 'fixed point'-lemma, which is the lemma that appears in the last part of the title: "... eine Verallgemeinerung des Brouwerschen Fixpunktsatzes". Von Neumann then derived the existence of a saddle point for ϕ as a direct consequence of this lemma. (See [von Neumann, 1937, p. 80].)

3.2. *The connection to the minimax theorem*

At this time von Neumann was fully aware of the connection between the game theoretical problem and the problem of existence of a solution to a system of linear inequalities:

> Die Lösbarkeit unseres Problems [The existence of a solution to the system (5)–(10)] hängt sonderbarerweise mit jener eines in der Theorie der Gesellschaftsspiele auftretenden Problems zusammen, das der Verf. anderwärtig behandelt hat ... Jenes Problem ist ein Specialfall von (*) und wird durch unsere Lösung von (*) auf eine neue Weise miterledigt.[24] [Von Neumann, 1937, p.79, note 2][25]

This is the first time von Neumann explicitly states that he has recognized a connection between the solution of systems of linear inequalities and the minimax solution of a two-person zero-sum game. The connection was not trivial. As I pointed out in the previous section, reading von Neumann's 1928 paper on its own terms without making recourse to later developments in game theory reveals no evidence that von Neumann had realized this connection to systems of linear inequalities in 1928. On the contrary his statement in 1937 that the question about solutions to the system of inequalities is "oddly" connected with the minimax solution shows that this was kind of unexpected. Had he already in 1928 been aware of this he would probably not have called it "odd" ten years later.

Regarding the fixed point technique used by von Neumann to show the existence of a saddle point in the 1937 paper it can be seen both from the title of the paper where the result is announced and from the following quote from the paper that von Neumann found it a quite important result which was interesting in itself

> Dieser verallgemeinerte Fixpunktssatz ... ist auch an sich von Interesse.[26] [Von Neumann, 1937, p. 73]

In the previous section I argued that von Neumann in 1928 probably was not aware of the fact that the existence of a saddle point could be proved on the basis of fixed point techniques. The above quotation and the fact that he found the generalised fixed point result so important that he announced it in the title indicate that he had not fully recognized this in 1928. If so he would probably have announced it at that time considering the importance he ascribed to it in 1937 and there is no mention of fixed point techniques what-so-ever in the 1928 paper. Another argument in favour of this is that he explicitly wrote in the 1937 paper that the game theoretic problem is solved in the 1937 paper in "a new way".

---

[24] "The question whether our problem has a solution is oddly connected with that of a problem occuring in the Theory of Games dealt with elsewhere. ... The problem there is a special case of (*) and is solved here in a new way through our solution of (*)." [von Neumann, 1937, (1945 p. 5, note 1)].

[25] The (*) in the above quotation refers to the saddle point formulation labelled (*) in Sect. 3.1.

[26] "This generalised fix-point theorem ... is also interesting in itself." [von Neumann, 1937 (1945 p. 1)].

4. The minimax theorem in the theory of convexity: The 1944 proof

As we have just seen, in 1937 von Neumann was aware of the connection between the minimax theorem and the solvability of systems of linear inequalities. The proof though was not built on the algebra of inequalities but was founded on topological methods. The first algebraic proof of the minimax theorem was due to the French mathematician Jean Ville who published it in the fourth volume (fascicule 2) of Émile Borel's treatise "Traité du calcul des probabilités et de ses applications" which appeared in 1938.

Borel himself had published a series of notes on games from 1921 to 1927. He was the first who tried to build a mathematical theory for games but after the publication of von Neumann's minimax theorem in 1928 he seems to have lost interest in the subject[27]. He published a note on game theory in 1927 and then he did not publish anything on games until this part of volume four of the treatise of probability came out 10 years later. In this volume Borel has a chapter written by himself devoted to game theory and quite strikingly there is no reference at all to von Neumann and the minimax theorem in that chapter.[28] Instead the minimax theorem is treated in a separate note by Jean Ville with the title "Théorème de M. von Neumann" [Ville, 1938].

Ville's algebraic proof is important because it exercised a direct influence on von Neumann during his work with the book "Theory of Games and Economic Behavior", which was the first collected and coherent book on game theory. Thereby Ville's proof gave rise to a development which led to the establishment of the minimax theorem in the theory of convexity.

I will only present the key tools in Ville's proof and not go into the details of the proof itself.[29] Ville derived his key tool as a corollary to the following lemma concerning linear forms, which he proved by induction:

> Let p linear forms in n variables be given:
>
> $$f_j(x) = \sum a_{ij} x_i \quad (j = 1, \dots, p;\ i = 1, \dots, n),$$
>
> where $x = (x_1, \dots, x_n)$.[30]
> Suppose they have the following property:[31]
>
> For all $x \geq 0$ there exists a j in $\{1, \dots, p\}$, such that $f_j(x) \geq 0$.
>
> Then there exists at least one set of nonnegative coefficients
>
> $$X_1, \dots, X_p \quad \text{with} \quad X_1 + \dots + X_p = 1,$$

---

[27] Before the work of Borel one only finds attempts to mathematize specific games like the card game "le Her" by James Waldegrave, baccarat by Joseph Bertrand in 1899 and chess by Ernst Zermelo in 1913. For accounts on these earlier attempts and on the work of Borel see [Dimand and Dimand, 1992]. For accounts on the work of Borel see also [Leonard, 1992].

[28] In [Leonard, 1992] Leonard discuss' this issue and concludes that "This can only be regarded as an act of deliberate omission by Borel." [Leonard, 1992, p. 46].

[29] See also [Leonard, 1992]. For further details on the proof see [Kjeldsen, 1999].

[30] Ville wrote $a_j{}^i$ instead of a_{ij}, I have changed the notation to make clear that i does not indicate an exponent.

[31] $x \geq 0$ means $x_i \geq 0$ for every $i = 1, \dots, n$.

such that

$$\sum_{j=1}^{p} X_j f_j(x) \geq 0 \quad \text{for all } x \geq 0.$$

[Ville, 1938, p. 105]

The key tool Ville was able to derive from this lemma was the following result

> Let $f_1, \ldots, f_p$ be p linear forms in n variables $x_1, \ldots, x_n$, and let ϕ be a linear form in the same variables. If for every point $x \geq 0$ at least one of the forms f_j assumes a value greater than or equal to the value of ϕ then a linear combination
>
> $$\psi = X_1 f_1 + \cdots + X_p f_p, \;\; X_j \geq 0, \;\; X_1 + \cdots + X_p = 1,$$
>
> exists for which $\psi \geq \phi$ for all $x \geq 0$. [Ville, 1938, p. 107]

From this result Ville gave a fairly easy proof of von Neumann's minimax theorem.

4.1. Von Neumann's 1944 proof

The final placement of game theory in general and the minimax theorem in particular within a context of linear inequalities and the theory of convexity was due to the joint work "Theory of Games and Economic Behavior" by von Neumann and the Austrian economist Oskar Morgenstern [von Neumann and Morgenstern, 1944]. According to Kuhn and Tucker the proof of the minimax theorem which von Neumann and Morgenstern presented in their book was inspired directly by the proof given by Ville:

> Oskar Morgenstern has told us [Kuhn and Tucker] that he drew Ville's article to von Neumann's attention after seeing it quite by chance while browsing in the library of the Institute for Advanced Study. They decided at once to adopt a similar elementary procedure, trying to make it as pictorial and simple to grasp as possible. [Kuhn and Tucker, 1958, p. 116]

How thrilled Morgenstern was when he discovered Ville's proof is evident from a note in his diary dated Christmas Eve 1941:

> Both [the 1938 book of Borel and the proof by Ville] are unknown to Johnny. Now he has discovered additional proofs that are becomming increasingly simple and are purely algebraic!! It necessitates some modification in the text, but we can print it. (Quoted in [Rellstab, 1992, p. 87])

The new proofs by von Neumann that Morgenstern speaks about were indeed very different from his earlier proofs. The proof they gave in "Theory of Games and Economic Behavior" is, as we shall see, of a purely algebraic nature and falls within what von Neumann and Morgenstern themselves characterised as

> the mathematico-geometrical theory of linearity and convexity. [von Neumann and Morgenstern, 1944, p. 128]

The theorem of the alternative for matrices

The essential tool in the proof of the minimax theorem that Morgenstern and von Neumann gave in 1944 is what they called "The Theorem of the Alternative for Matrices":

> If A is a $(n \times m)$ matrix then exactly one of the following two systems of inequalities has a solution:
>
> $$Ax \leq 0, \quad x \geq 0, \quad \sum_{j=1}^{m} x_j = 1,$$
>
> $$wA > 0, \quad w > 0, \quad \sum_{i=1}^{n} w_i = 1.$$
>
> [von Neumann and Morgenstern, 1944, p. 138–141]

They derived this theorem as a direct consequence of the theorem of supporting hyperplanes which states that

> Given $x_1, \ldots, x_p$ in $\mathbf{R}^n$. Then a y in $\mathbf{R}^n$ either belongs to the convex set C spanned by $x_1, \ldots, x_p$, or there exists a hyperplane which contains y such that C falls entirely within one half-space produced by that hyperplane. [von Neumann and Morgenstern, 1944, p. 134]

In order to use this theorem to prove "The Theorem of the Alternative for Matrices" von Neumann and Morgenstern considered an $(n \times m)$ matrix, A, with elements $a(i, j)$, $i = 1, \ldots, n$; $j = 1, \ldots, m$. They formed the convex set C spanned by the m column vectors in A together with the n unit vectors in $\mathbf{R}^n$. Putting $y = 0$, either 0 belongs to C or to a hyperplane H, such that all of C is contained in one half-space produced by that hyperplane [von Neumann and Morgenstern, 1944, p. 139]. In the first case they could prove the existence of an x in $\mathbf{R}^m$ for which $x_1 \geq 0, \ldots, x_m \geq 0$, $\sum_{j=1}^{m} x_j = 1$, and such that the inequalities

$$\sum_{j=1}^{m} a(i, j)x_j \leq 0$$

are satisfied for $i = 1, \ldots, n$. In the second case, that is where 0 does not belong to C, they showed the existence of a vector w in $\mathbf{R}^n$ with $w_1 > 0, \ldots, w_n > 0$, $\sum_{i=1}^{n} w_i = 1$, such that the following inequalities are satisfied:

$$\sum_{i=1}^{n} a(i, j)w_i > 0 \quad \text{for} \quad j = 1, \ldots, m.$$

These two possibilities, or alternatives as von Neumann and Morgenstern called them, exclude each other, and this finished their proof of "The Theorem of the Alternative for Matrices".

From this result they proved the minimax theorem for two-person zero-sum games in the following way: Keeping the notation from von Neumann's 1928 paper (see (1), in Sect. 2.2) they let

$$h(\xi, \eta) = \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g(p, q)\xi_p \eta_q$$

be the expected value for the player S_1. By defining A to be the matrix $(g(p, q))_{(\Sigma_1 \times \Sigma_2)}$, they obtained either the existence of a vector $\xi \in \mathbf{R}^{\Sigma_1}$ for which $\xi \geq 0$, $\sum \xi_p = 1$, such that

$$\sum_{p=1}^{\Sigma_1} g(p, q)\xi_p \geq 0 \quad \text{for} \quad q = 1, \dots, \Sigma_2, \tag{11}$$

or the existence of a vector $\eta \in \mathbf{R}^{\Sigma_2}$ for which $\eta \geq 0$, $\sum \eta_q = 1$, such that

$$\sum_{q=1}^{\Sigma_2} g(p, q)\eta_q \leq 0 \quad \text{for} \quad p = 1, \dots, \Sigma_1. \tag{12}$$

If (11) holds true then

$$v_1 = \max_\xi \min_\eta h(\xi, \eta) \geq 0.$$

If on the other hand (12) holds true then

$$v_2 = \min_\eta \max_\xi h(\xi, \eta) \leq 0.$$

From this von Neumann and Morgenstern concluded that

$$\text{either} \quad v_1 \geq 0 \quad \text{or} \quad v_2 \leq 0$$

that is *never*

$$v_1 < 0 < v_2. \tag{13}$$

The final step in the proof was to show that (13) cannot be the case, and even that, for any number w, the inequality

$$v_1 < w < v_2$$

cannot be satisfied. Since $v_1 \leq v_2$ is always true von Neumann and Morgenstern had proven the equality:

$$v_1 = \max_\xi \min_\eta h(\xi, \eta) = \min_\eta \max_\xi h(\xi, \eta) = v_2.$$

In this way von Neumann and Morgenstern in 1944 reduced the proof for the minimax theorem to a fairly simple consequence of the theorem of “Alternatives for Matrices” which is a purely algebraic theorem with in the theory of systems of linear inequalities.

5. Conclusion on von Neumann's perception of the minimax theorem

The above analysis of the development of von Neumann's understanding of the different mathematical contexts in which the minimax theorem presented itself during the period 1928 to 1944, clearly shows that his recognition of the connections between the minimax theorem on the one hand, and fixed point results and the theory of linear inequalties on the other hand, only emerged gradually. The full understanding of the connection to fixed point theorems was not present until 1937 while the final establishment and realization of the minimax theorem as a result belonging to the theory of linear inequalities and the theory of convexity was not fully recognized until 1944.

The history told until now can also serve as an illustration of how mathematics evolves. In this case a problem (the question of the existence of optimal solutions to two-person zero-sum games) emerged in connection with a new kind of mathematical questions (the mathematization of games). The problem is solved, and in the beginning possible connections to other branches of mathematics can be very difficult, if not impossible, to realize. Later the problem or a similar problem crops up again in another context (the economic model of von Neumann), one recognizes the connection and the complexity of the problem diminishes, the underlying structure of the proof becomes visible (fixed point techniques) and new general results (the extension of Brouwer's fixed point theorem) emerge, which are interesting in themselves and not limited to the context in which they originally were derived. Finally the problem is recognized to be a simple consequence of fundamental theorems in a different branch of mathematics (theories of linear inequality and convexity).

6. The discussion of priority: the significance of the minimax theorem

The 1928 paper by von Neumann was generally taken to mark the beginning of game theory. Borel's earlier notes on the subject from the beginning of the twenties were not generally known until after 1953 when the French mathematician Maurice Fréchet had three of them translated into English. In the introduction to the translation Fréchet argued for the importance of Borel's work:

> It was only relatively recently that I began to occupy myself with the theory of probability and its applications, which explains why the notes that Émile Borel . . . published between 1921 and 1927 on the theory of psychological games escaped my attention. It was chance to begin with . . . because, in the extensive literature devoted to this theory [game theory] and its applications in recent years, references to earlier work do not lead back, in general, further than to the important paper published in 1928 by Professor von Neumann. But, in reading these notes of Borel's I discovered that in this domain, as in so many others, Borel had been an initiator. [Fréchet, 1953a, p. 95]

In order to understand the priority debate and how it connects to the significance of the context to be discussed below, we need to know a little about Borel's work.

6.1. Borel's work on game theory

The first of Borel's notes on the subject was published in 1921 [Borel, 1921]. He considered only symmetric games with two players A and B. He introduced the concept "méthode de jeu" (method of play) and the fundamental question asked by Borel was then, whether is was possible to determine a "méthode de jeu meilleure" (best method of play). It was not quite clear what was to be understood by a "méthode meilleure" but his concept of "méthode de jeu" was the same as von Neumann's, that is, what now is called a pure strategy [Borel, 1921, p. 1304]. Like von Neumann, Borel assumed the players had a finite number of strategies $C_1, \ldots, C_n$ to choose from.

Borel's inspiration to investigate games came from his work on probability and in his first paper he was looking for the probabilities of winning the game. His starting point was, that if A chooses the strategy C_i, and B chooses the strategy C_k, then the probability a, that A wins the game, can be calculated. The probability for player B is then $b = 1 - a$. To indicate that these probabilities are dependent of the choices of strategies he put

$$a = \frac{1}{2} + \alpha_{ik} \tag{14}$$

and

$$b = \frac{1}{2} + \alpha_{ki} \tag{15}$$

where α_{ik} and α_{ki} lie between $-\frac{1}{2}$ and $+\frac{1}{2}$ and satisfy the relation

$$\alpha_{ik} + \alpha_{ki} = 0.$$

Like von Neumann he also considered the concept of what later became known as mixed strategies [Borel, 1921, p. 1305]. But in contrast to von Neumann, who considered the actions of both players simultaneously, Borel began by examining singular cases, calculating if it would be possible for one of the players to choose a mixed strategy such that the probability that he or she would win would be equal to $\frac{1}{2}$, no matter what strategy the other player would choose. In the 1921 note he calculated the case where there are only three pure strategies to choose from and he reached a positive conclusion. In general, though, he was convinced that for games with more than three pure strategies the answer would be negative [Borel, 1921, p. 1306]. Two years later he had done the calculations for games with five pure strategies which shows that the answer also in this case turned out to be positive and he thought that it would probably also be true for seven pure strategies, but he still thought that for a larger number of strategies the answer should be no [Borel, 1923, p. 1117].[32]

[32] Borel only did the calculations for symmetric games with an uneven number of pure strategies. The reason for this is the appearence, in the calculations of the (mathematical) expected value of a symmetric game, of a skew symmetric matrix of the same order as the number of pure strategies. Thus, if the number of pure strategies is an uneven number, it becomes especially nice, because the determinant of the corresponding skew symmetric matrix will then be equal to zero. (see [Borel, 1921, p. 1306]).

In 1924 Borel included a chapter on games in his book on probability [Borel, 1924]. Instead of looking at the probabilities for winning as he had done in 1921, he now let α_{ik} denote an amount of money, which player B has to give to player A, if player A chooses strategy C_i and player B chooses strategy C_k. The question he was trying to answer then was, is it possible for player A to choose a mixed strategy such that the expected value he or she can get is 0, no matter which mixed strategy player B chooses? That is, can player A choose a strategy that in all cases can protect A from loosing money? This is in principle the same question that led von Neumann to the minimax theorem namely, what is the best you can do in the worst possible case, which is the case where your opponent somehow has gained knowledge about your choice of strategy.

Changing the interpretation of the α_{ik}'s from being part of the probability a (14), that A wins, to being the amount of money B has to give to A, if they choose the strategies C_k and C_i respectively, did not induce Borel to change his mind about the answer to the question under consideration. He still believed that if the number of pure strategies was larger than seven, the answer would be no. Two years later he had not yet found an argument for his belief, which follows from a note published in 1926, where he discussed the question again. Only this time he formulated both situations: Is it always possible for player A to choose a mixed strategy for which the expected value of the game will be zero no matter what strategy player B chooses, or is this not the case? And thereby left it as an open question [Borel, 1926]. The second situation contradicts the minimax theorem. The fact that he formulated the positive situation first has been interpreted as implying that he seriously doubted his original views and was beginning to believe that maybe what is now called the minimax theorem would turn out to be true.[33]

The reason why Borel at the outset did not belive in a positive answer has been discussed by Luca Dell'Aglio in the paper "Divergences in the History of Mathematics: Borel, von Neumann and the Genesis of Game Theory" [Dell'Aglio, 1995]. Dell'Aglio argues that Borel had a psychological interpretation of the concept of mixed strategies, which

> . . . constitute the conceptual basis of Borel's negation of the minimax theorem in his earlier research into game theory. [Dell'Aglio, 1995, p. 21]

The psychological interpretation enters the picture because Borel on several occasions talks about the advantage of being a better psychologist. The player who is a better observer and analyst than the opponent will have an advantage in the game, which is not true for optimal solutions covered by the minimax theorem. Dell'Aglio concludes that

> . . . the divergence over the validity of the minimax theorem was ultimately due to a difference in the conceptual and technical structure underlying the two theories. In other words, Borel and von Neumann produced different theoretical forecasts because they were working on different basic problems. [Dell'Aglio, 1995, p. 40]

The two different problems that Dell'Aglio is referring to emerge because von Neumann's point of departure was "the possibility of the existence of equilibria in games played by equal players" [Dell'Aglio, 1995, p. 40] while "Borel took into consideration a

---

[33] See e.g. [Fréchet, 1953b, p. 122].

similar problem but supposing one player has acquainted himself with the psychological characteristics of his opponent" [Dell'Aglio, 1995, p. 40].

I do not quite understand in what sense Dell'Aglio meant that von Neumann and Borel are studying two different problems. Both of them had as point of departure that you choose your strategy without knowing what your opponent is going to choose. In Borel's calculations for the cases with three and five pure strategies he is, like von Neumann, looking for a strategy that can protect you from being in a losing position, no matter what the other player does. For such a strategy the result of the game will not change to your disadvantage even if your opponent somehow found out which one you picked. In a situation like that it does not matter which one is a better psychologist. As far as I can see the main difference between their work is the approach in their investigations. Borel did not consider the mixed strategies of the two players simultaneously. He did not contemplate the interaction between the two players' independent and simultaneous choice of strategy. Von Neumann did so and that brought the various 'minmax' and 'maxmin' considerations into the picture and it is precisly the interaction between these that made him realize the solution as a saddle point.

6.2. Discussion of priority

In the quotation cited at the beginning of section 6 from the introduction by Fréchet to the translation of the notes of Borel, Fréchet announced that Borel had been an "initiator" in the domain of game theory. In a commentary[34] Fréchet argued for this opinion:

> Borel was the first to indicate the potential importance for this theory of knowing whether this theorem [the minimax theorem], applied to n manners of playing, is true for arbitrary n. He did, moreover, demonstrate it for $n = 3$ and $n = 5$, but only for these values. [Fréchet, 1953b, p. 122]

This introduction and commentary of Fréchet caused a brief priority discussion between von Neumann and Fréchet. According to L. J. Savage, the translator of Borel's work, von Neumann got very angry when he learned what Fréchet had written [Heims, 1980, p. 440, note 14]. In "Communication on the Borel Notes" von Neumann acknowledged that Borel had been the first one to introduce the concepts of pure and mixed strategies but, he continued,

> The relevance of this concept [of mixed strategies] in his [Borel's] hands was essentially reduced by his failure to prove the decisive 'minimax theorem', or even to surmise its correctness. As far as I can see, there could be no theory of games on these bases without that theorem. . . . I felt that there was nothing worth publishing until the 'minimax theorem' was proved. [von Neumann, 1953, p. 124–125]

[34] The translation of the notes of Borel was published in *Econometrica* in 1953 together with an introduction by Fréchet. The translation was followed first by "Commentary on the Three Notes of Emile Borel" also by Fréchet and second by "Communication on the Borel Notes" by von Neumann. All of it was published together with the translation of the notes of Borel.

What I find interesting is not so much the priority debate in itself but more the following remark by Fréchet, in the "Commentary on the Three Notes of Emile Borel" which shows a view very different from that of von Neumann on the importance of the minimax theorem:

> Again, it may be mentioned, that even if Borel had, before von Neumann, established the minimax theorem in its full generality; the profound originality of Borel's notes would not have been augmented nor even touched from the economic point of view. He would not thereby have even enriched the set of properly mathematical discoveries for which Borel has acquired a world-wide reputation. He would have, like von Neumann, simply entered an open door. . . . the same theorem and even more general theorems had been independently demonstrated by several authors well before the notes of Borel and the first paper of von Neumann. [Fréchet, 1953b, p. 122]

The proofs Fréchet is referring to are proofs of theorems similar to von Neumann and Morgenstern's "Alternatives for Matrices", i.e., theorems about solutions to systems of linear inequalities by Minkowski, Farkas, Stiemke, and Weyl.[35]

In 1953 the minimax theorem was realized to be a simple consequence of those classical theorems about solutions to systems of linear inequalities, but von Neumann derived the minimax theorem in a theory of "Gesellsschaftsspiele" which was a completely different mathematical context. The techniques used by von Neumann in 1928 had at first nothing to do with linear inequalities, and it was not until Ville's proof in 1938 that this connection was recognized, a connection von Neumann and Morgenstern then developed further in their book of 1944. But as von Neumann's 1928 proof and his 1937 proof clearly demonstrate and as he himself wrote in 1953 in his answer to Fréchet:

> This connection may now seem very obvious to someone who first saw the theory after it had obtained its present form. (O. Morgenstern and myself, in our presentation in 1943, made, for didactical reasons, every effort to emphasize this connection.) However, this was not at all the aspect of the matter in 1921–1938. The theorem, and its relation to the theory of convex sets were far from being obvious . . .. It is common and tempting fallacy to view the later steps in a mathematical evolution as much more obvious and cogent after the fact than they were beforehand. [von Neumann, 1953, p. 125]

7. Conclusion on the significance of the context

In this discussion Fréchet advocates the point of view, that the significance of a mathematical theorem is independent of the mathematical context in which it was derived. The history of von Neumann's development and conception of the minimax theorem shows that it was far from being trivial and took a larger effort to realize the connection between solutions of systems of linear inequalities and the existence of optimal strategies for two-person zero-sum games. The fact that the minimax theorem later turned out to be a simple consequence of theorems of inequalities proved earlier, does not

[35] See [Farkas, 1901, p. 5–7], [Stiemke, 1915, p. 340], [Gordan, 1873, p. 23–28], [Minkowski, 1896, p. 39–45], [Weyl, 1935].

render the minimax theorem superfluous or worthless in relation to the development of mathematics, as Fréchet seems to imply. In his assessment whether the minimax theorem has "enriched the set of properly mathematical discoveries" or not, an evaluation of the significance of the theorem for the developing of new mathematics is lacking. The mathematical context in which a result is derived determines its formulation and interpretation and thereby also which kind of new research it can lead to. The questions that guide research in game theory are not necessarily the same as those guiding research in the abstract theory of linear inequalities. Hence, the minimax theorem can, from a game-theoretic point of view, be very different from the theorems of linear inequalities.

The minimax theorem of von Neumann had a tremendeous influence on the further development of game theory which became a very active field of research after World War II.[36] It also had a decisive influence on the development of some new disciplines in applied mathematics, especially linear and nonlinear programming which originated in connection with the Second World War.[37]

References

Borel, É. (1921): "La théorie du jeu et les équations intégrales à noyau symétrique." *Comptes Rendus de l'Académie des Sciences*, dec. 19, 173, 1921, pp. 1304–1308.

Borel, É. (1923): "Sur les jeux où le hasard se combine avec l'habileté des joueurs." *Comptes Rendus de l'Académie des Sciences*, dec. 26, 1923, pp. 1117–1118.

Borel, É. (1924): "Sur les jeux où interviennent le hasard et l'habileté des joueurs." In É. Borel: *Théorie des Probabilités.* Paris: Librairie Scientifique, J. Hermann, 1924, pp. 204–224.

Borel, É. (1926): "Un théorème sur les systèmes de formes linéaires à déterminant symétrique gauche." *Comptes Rendus de l'Académie des Sciences*, 183, nov. 22, 1926, pp. 925–927.

Borel, É. et al. (1938): *Traité du Calcul des probabilités et de ses applications.* Tome 4, Fasc. 2, Applications aux Jeux de Hasard. Paris: Gauthier-Villars, 1938.

Dell'Aglio, L. (1995): "Divergences in the History of Mathematics: Borel, von Neumann and the Genesis of Game Theory." *Rivista di Storia della Scienza*, 3, 1995, pp. 1–46.

Dimand, R. W. and Dimand, M. A. (1992): "The Early History of the Theory of Strategic Games from Waldegrave to Borel." In E. R. Weintraub (ed.) *Towards a History of Game Theory.* Durham and London: Duke University Press, 1992, pp. 15–28.

Farkas, J. (1901): "Theorie der einfachen Ungleichungen." *Journal für die reine und angewandte Mathematik*, 124, 1901, pp. 1–27.

Fréchet, M. (1953a): "Emile Borel, Initiator of the Theory of Psychological Games and its Application." *Econometrica*, 21, 1953, pp. 95–96.

Fréchet, M. (1953b): "Commentary on the three notes of Emile Borel." *Econometrica*, 21, 1953, pp. 118–124.

Gordan, P. (1873): "Über die Auflösungen linearer Gleichungen mit reelen Coefficienten." *Mathematische Annalen*, 6, 1873, pp. 23–28.

Heims, S. J. (1980): *John von Neumann and Norbert Wiener.* Cambridge, Massachusetts: The MIT Press, 1980.

Ingrao, B. and Israel, G. (1990): *The Invisible Hand.* Cambridge, Massachusetts: The MIT Press, 1990.

[36] See for example [Mirowski, 1991].
[37] See [Kjeldsen, 1999, 2000a, 2000b].

Kjeldsen, T. H. (1999): "A Contextualised Mathematico-historical Analysis of Nonlinear Programming: Development and Multiple Discovery." (in Danish), IMFUFA, Text 372, Roskilde University, 1999.

Kjeldsen, T. H. (2000a): "The Emergence of Nonlinear Programming: Interactions between Practical Mathematics and Mathematics Proper." *The Mathematical Intelligencer*, Vol. 22, No. 3, Summer 2000, pp. 50–54.

Kjeldsen, T. H. (2000b): "A Contextualized Historical Analysis of the Kuhn-Tucker Theorem in Nonlinear Programming: The Impact of World War II." *Historia Mathematica*, Vol. 27, No. 4, Nov 2000, pp. 331–361.

Kuhn, H. W. and Tucker, A. W. (1958): "John von Neumann's Work in the Theory of Games and Mathematical Economics." *Bulletin of the American Mathematical Society*, 64, 1958, pp. 100–122.

Leonard, R. J. (1992): "Creating a Context for Game Theory." In E. Roy Weintraub (ed.): *Towards a History of Game Theory.* Durham and London: Duke University Press, 1992, pp. 29–76.

Leonard, R. J. (1995): "From Parlor Games to Social Science: von Neumann, Morgenstern, and the Creation of Game Theory 1928–1944." *The Journal of Economic Literature*, 33, 1995, pp. 730–761.

Minkowski, H. (1896): *Geometrie der Zahlen.* Leipzig: B. G. Teubner, 1896.

Mirowski, P. (1991): "When Games Grow Deadly Serious: The Military Influence on the Evolution of Game Theory." In D. G. Goodwin (ed.): *Economics and National Security*, Annual Supplement to Volume 23, History of Political Economy. Durham and London: Duke University Press, 1991, pp. 227–255.

Mirowski, P. (1992): "What Were von Neumann and Morgenstern Trying to Accomplish?" In E. Roy Weintraub (ed.): *Toward a History of Game Theory.* Durham and London: Duke University Press, 1992, pp. 113–147.

Rellstab, U. (1992): "New Insights into the Collaboration between John von Neumann and Oskar Morgenstern on the *Theory of Games and Economic Behavior*." In E. Roy Weintraub (ed.): *Towards a History of Game Theory.* Durham and London: Duke University Press, 1992, pp. 77–94.

Stiemke, E. (1915): "Über positive Lösungen homogener linearer Gleichungen." *Mathematische Annalen*, 76, 1915, pp. 340–342.

Ulam, S. (1958): "John von Neumann, 1903–1957." *Bulletin of the American Mathematical Society*, 64, 1958, pp. 1–49.

Ville, J. (1938): *Sur la Théorie génerale des jeux où intervient l'habileté des joueurs.* In É. Borel et al. (eds.:) *Traité du Calcul des probabilités et de ses applications.* Paris: Gauthier-Villars, 1938, pp. 105–117.

von Neumann, J. (1928a): "Sur la théorie des jeux." *Comptes Rendus de l'Académie des Sciences*, 186.25 (18. of June), 1928, pp. 1689–1691.

von Neumann, J. (1928): "Zur Theorie der Gesellschaftsspiele." *Mathematische Annalen*, 100, 1928, pp. 295–320.

von Neumann, J. (1937): "Über ein ökonomische Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes." In K. Menger (ed.) *Ergebnisse eines Mathematischen Kolloquiums.* Wien, 1937, pp. 73–83.

von Neumann, J. (1945): "A Model of General Economic Equilibrium." (English translation of [von Neumann, 1937]) in *Review of Economic studies*, vol. 13, 1945, pp. 1–9.

von Neumann, J. (1953): "Communication on the Borel Notes." *Econometrica*, 21, 1953, pp. 124–125.

von Neumann, J. (1959): "On the Theory of Games of Strategy." (English translation of [von Neumann, 1928]) in A. W. Tucker and R. D. Luce (eds.:) *Contributions to the Theory of Games.* Princeton, New Jersey: Princeton University Press, 1959, pp. 13–42.

von Neumann, J. (1963): *Collected Works*, (ed.:) A. H. Taub, Pergamon Press: New York, 1963.
von Neumann, J. and Morgenstern, O. (1944): *Theory of Games and Economic Behavior.* Princeton: Princeton University Press, Princeton, 1944.
Weyl, H. (1935): "Elementare Theorie der konvexen Polyeder." *Mathematici Helvetici*, 7, 1935, pp. 209–306.

Department of Mathematics
Roskilde University
P.O. Box 260
4000 Roskilde
Denmark
thk@ruc.dk

(Received March 1, 2001)