PRACTICE PROBLEMS FOR THE MATH 3C FINAL

REMEMBER: NO CALCULATORS, CLOSED BOOK, NO NOTES

UPDATES: 7:53 am March 12: Edits in prbs 19, 20, 21.
10:25 pm on March 13: Edit in prb 7.
11:48 am on March 15: Edit in prb 12.
6:40 pm on March 15: Edits in "About" prb 1 and 19.

1. Consider the surface $V = f(u, w) = w + w^2 \cos(uw)$ in (u, w, V) space.
   (a) Draw the w-section $w = 1$ (do this in 2 dimensions, not 3). Label your axes and label a couple of points on the curve with their coordinates.
   (b) Find the equation of the tangent plane at the point $(u, w) = (\pi, 1)$. Your answer should be in the form $V =$ a linear function of u and w.
2. Consider the differential equation $y' = 2ty + 1$. In the same picture:
   (a) Sketch some isoclines.
   (b) Sketch the direction field.
3. Label each of the following differential equations as S (separable), LH (linear and homogeneous), LI (linear and inhomogeneous), A (autonomous), or N (none of the preceding). Use all labels that apply.
   (a) $y' = e^{\sin(t)}y - t$
   (b) $dw/ds + \cos(w^2)e^s = 0$.
   (c) $y' = (y - t)^2$.
   (d) $z' - (e^t - e^{-t})z = 0$.
   (e) $y' = e^y + t$.
   (f) $y' = (1 + t^3)/(4 + \cos(y)^2)$.
   (g) $y' = y^3 + e^y + 17$.
4. Determine all the equilibrium solutions of the differential equation

$$y' = (y + 1)y^2(y - 3)^3(y - 4)^2.$$

   Sketch them in a direction field picture for the equation and mark them on the phase line (the line $t = 0$) with an open circle if they are unstable, a filled circle if they are stable and a split circle if they are semistable. Your direction field picture should show why these markings are correct.
5. Consider the differential equation $y' - t\sin(t)y^4 = 0$
   (a) Find all equilibrium solutions.
   (b) Find the general solution. Verify that your general solution does solve the equation.
   (c) Find the solution satisfying $y(\pi) = 100$.
6. Use $h = .25$ to perform four steps of the Euler method to approximate the solution of the initial value problem

$$y' = y, \qquad y(0) = 1.$$

   at $t = 1$. The exact value of $y(t)$ at $t = 1$ is $e = 2.7182818285 - --$, what is the total error of your approximation?

7. Consider the differential equation $y' = \cos(t)y + t$
   (a) Verify that $y(t) = e^{\sin(t)} \int_{23}^{t} e^{-\sin(x)} x\, dx$ is solution to the differential equation with $y(23) = 0$. (NOTE: You are not asked to solve the equation and you should not try to explicitly evaluate the integral. You are asked instead to check that the given function satisfies the equation and the given initial condition).
   (b) Find the general solution of the equation. (Hint: You already have a "particular solution".)
   (c) Find the solution which satisfies $y(23) = 115$.
8. Find, by inspection, a particular solution of the equation $y' + 8\frac{1}{1+t}y = \frac{1}{1+t}$. Then find the general solution of the equation.
9. Let $Ly = y' + p(t)y$ where $p(t)$ is some function. Assume that $Ly_1 = t$, $Ly_2 = \cos(t)$, and $Ly_3 = 0$ where y_1, y_2 and y_3 are some functions and $y_3 \neq 0$. What is the general solution of $Ly = 2t - 3\cos(t)$?
10. Find the solution of the IVP
$$\frac{dy}{dt} = \sin(t)y + \sin(t)\cos(t), \quad y(1) = 1.$$
Your final answer should be in closed form and contain no unevaluated integrals.
11. Find the solution of the IVP
$$\frac{dy}{dt} + 4\frac{y}{t} = \frac{e^t}{t^3}, \quad y(1) = 1.$$
Your final answer should be in closed form and contain no unevaluated integrals.
12. The solution of $w' + q(t)w = h(t)$, $w(a) = w_0$ is
$$w(t) = e^{-\int_a^t q(r)\, dr} w_0 + e^{-\int_a^t q(r)\, dr} \int_a^t e^{\int_a^s q(r)\, dr} h(s)\, ds.$$
   (a) Derive this formula.
   (b) Show directly that the formula provides a function which satisfies the equation and the initial condition.
13. Suppose the function y satisfies
$$\int_0^{y(t)} \frac{1}{g(s)}\, ds = \int_0^t f(s)\, ds$$
for all t in some interval. Show, using only the Fundamental Theorem of Calculus and the Chain Rule, that then $y'(t) = g(y(t))f(t)$; that is y solves $y' = g(y)f(t)$. Justify any computations by giving the reasons why you think they are correct.
14. Let $x(t)$ be the mass of a radioactive substance at time t.
   (a) The rate of decay of a radioactive substance is proportional to the amount present at any time. Use this information to find the differential equation that $x(t)$ satisfies.
   (b) Let's say that initially there is a 25 g of radioactive material and it is decaying continuously at a rate of 2%/year. Solve this IVP for x.
   (c) What is the half-life of this substance? (You may leave logs in the answer.)

15. Al Capone is blending moonshine in the basement of his secret bar. His apparatus consists of two connected barrels. Newly-distilled whiskey containing 80% alcohol is dripping into barrel 1 at a rate of 10 gal/h. This mixture is constantly stirred and flows out of barrel 1 and into barrel 2 at the same rate. The mixture in barrel 2 is also thoroughly stirred and flows out at the same rate to be bottled. Barrel 1 is 50 gals, barrel 2 is 100 gals. They are initially both filled with old whiskey containing 40% alcohol.

    (a) Set up an IVP for the amount of pure alcohol in barrel 1 as a function of time. Solve this IVP.
    (b) Do the same for barrel 2.
    (c) How long before the whiskey flowing out of barrel 2 contains 50% alcohol?

16. Lt. Columbo is investigating a bank robbery, which took place at 4PM. He visits the suspect's home at 8PM. The suspect's friends assure him that they (incl. the suspect) were watching the Smurfs on video all day long, and they didn't even leave the house since breakfast. On his way out, Columbo notices that the suspect's car feels warm. He measures the temperature of the coolant and it's 25°C. An hour later, he sneaks back to take another measurement. It is now 20°C. The thermostat in the garage is set to 15°C. He knows that the normal coolant temperature in a running engine is 95°C.

    (a) Assuming that Newton's Law of Cooling applies, write a differential equation to describe the temperature of the coolant in the car.
    (b) Solve the equation, and use the given data to figure out the constants in your solution.
    (c) Should Columbo believe the suspect and his friends?

17. A barrel of monkeys is released on a monkey-less tropical island. Five years later, 280 monkeys are found swinging in the trees. Another 5 years later, 700 monkeys are counted on the island. Assume that the monkey population obeys the logistic equation and the island's banana crop can sustain 2800 monkeys.

    (a) Write down the general logistic equation and solve it. Show all details of how you solve it.
    (b) Using the population data and the maximum population, find the constants you need to use in the logistic equation for the monkey population. Do not make any approximations. (Hint: You can take advantage of the fact that the equation is autonomous to make the initial conditions simple.)
    (c) How many monkeys were in the original barrel?

18. (a) Find the Taylor series for $f(x) = 1/x$ at $x = 1$.
    (b) What is the radius of convergence?

19. Your professor finds it convenient to define a new function for his research. This function he names UCSB in honor of his institution. It is given by

$$\mathrm{UCSB}(x) = \frac{\sin(x^2) - x^2}{x^3}.$$

    (a) Find the Taylor series for UCSB about the origin. (Hint: you may use what you know about the Taylor series for $\sin(x)$.)

(b) The prof names the anitderivative of UCSB which is 5 at $x = 0$ UNIV. Use

$$\frac{d}{dx}\mathrm{UNIV}(x) = \mathrm{UCSB}(x), \quad \mathrm{UNIV}(0) = 5,$$

and the result from part (a) to find the Taylor series for UNIV(x) at $x = 0$.

(c) Find $\mathrm{UNIV}^{(6)}(0)$ and $\mathrm{UNIV}^{(8)}(0)$.

(d) Approximate the value UNIV(.12) by the Taylor polynomial of degree 6.

20. Show the following:

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges. (Hint: the derivative of $\ln(\ln(x))$ is $1/(x\ln(x))$.)

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)^2}$ converges. (Hint: the derivative of $1/\ln(x)$ is $-1/(x\ln(x)^2)$.)

21. Determine the values of x for which the following power series converge:

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{n\,2^n}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$.

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2}(1-x)^n$.

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n\, n!\,(x+5)^{2n-1}$.

Some Solutions and Comments

Warning: "Learning" the answers to practice problems is nearly - probably completely - useless. If you know how to do all the problems above in the sense that you can do all problems of the genres they represent, no matter what notation is used in the statements and no matter how they are formatted, you are in quite good shape. The problems on the final will not be exactly the same as the ones above, but they will be doable with the techniques and ideas needed to solve the ones above. Some of these are more demanding than a version on the final might be. Also, the final will not be this long nor this comprehensive. Hey, we are always teaching!

About Problem 1(b). We must compute the partial derivatives of f. We have $f_u(u,w) = -w^3\sin(uw)$ and $f_w(u,w) = 1 - w^2u\sin(uw) + 2w\cos(uw)$. Since $f(\pi,1) = 0$, $f_u(\pi,1) = 0$, and $f_w(\pi, 1) = -1$, the tangent plane at $(\pi, 1)$ is given by

$$V = 0 + 0(u - \pi) - (w - 1) = -(w - 1).$$

About Problem 3. (a) LI, (b) S, (c) N, (d) LH,S (e) N, (f) S, (g) S, A.

About Problem 4. The equilibrium solutions are $y = -1$ (stable), $y = 0$ (semistable), $y = 3$ (unstable) and $y = 4$ (semistable).

About Problem 5. (a) $y = 0$, (b) $y = (3(t\cos(t) - \sin(t)) + c)^{-1/3}$: by the the power and chain rules:

$$\begin{aligned} y' &= -\frac{1}{3}(3(t\cos(t) - \sin(t)) + c)^{-4/3}(-3t\sin(t)) \\ &= y^4(t\sin(t)). \end{aligned}$$

(c) $c = 100^{-3} + 3\pi$ in (b).

About Problem 6. You should have $y_4 = 2.4414062500 - -$ and your total error is then $2.7182818285 - 2.4414062500$ (good enough for government work).

About Problem 7. (a) It is clear that $y(23) = 0$ since the $\int_{23}^{23}$ of anything is 0. By the product rule, the chain rule and the fundamental theorem of calculus, if $y(t) = e^{\sin(t)} \int_{23}^{t} e^{-\sin(x)} x\, dx$, then

$$\begin{aligned} \frac{d}{dt}y(t) &= \cos(t)e^{\sin(t)} \int_{23}^{t} e^{-\sin(x)} x\, dx + e^{\sin(t)} \frac{d}{dt} \int_{23}^{t} e^{-\sin(x)} x\, dx \\ &= \cos(t)e^{\sin(t)} \int_{23}^{t} e^{-\sin(x)} x\, dx + e^{\sin(t)} e^{-\sin(t)} t \\ &= \cos(t)y + t. \end{aligned}$$

(b) By separation of variables or direct verification, $e^{\sin(t)}$ is a solution of the homogeneous equation. Thus

$$y = ce^{\sin(t)} + e^{\sin(t)} \int_{23}^{t} e^{-\sin(x)} x\, dx$$

is the sum of a particular solution and the general solution of the homogeneous equation, hence it is the general solution. Imposing the initial condition $y(23) = 115$ leads to $c = 115/e^{\sin(23)}$.

About Problem 8. The easiest solutions to find by inspection are constant solutions, when there are any. Here $y = 1/8$ works. The homogeneous equation is solved by $(1+t)^{-8}$ and the general solution is $y = c(1+t)^{-8} + 1/8$.

About Problem 9. Since L is linear,

$$L(2y_1 - 3y_2 + cy_3) = 2Ly_1 - 3Ly_2 + cLy_3 = 2t - 3\cos(t)$$

and so $y = 2y_1 - 3y_2 + cy_3$ is the general solution.

About Problem 10. By any method one finds the general solution $y(t) = 1 - \cos(t) + ce^{-\cos(t)}$ and then $c = \cos(1)e^{-\cos(1)}$ yields $y(1) = 1$. The integral needed here, $\int \sin(t)\cos(t)e^{\cos(t)}\,dt = (1-\cos(t))e^{\cos(t)} + c$, is done by parts in conjunction with substitutions.

About Problem 11. By any method one finds the general solution

$$y(t) = \frac{(t-1)e^t}{t^4} + \frac{c}{t^4}$$

and then $c = 1$ yields $y(1) = 1$. The integral needed here, $\int te^t\,dt = (t-1)e^t + c$, is done by parts.

About Problem 13. Let $H(r) = \int_0^r (1/g(s))\,ds$. Then $H'(r) = 1/g(r)$ by the Fundamental Theorem of Calculus. Given $H(y(t)) = \int_0^t f(s)\,ds$, differentiating both sides and using the chain rule and the Fundamental Theorem again yields $y'(t)H'(y(t)) = f(t)$ or $y'(t)/g(y(t)) = f(t)$.

About Problem 16. (c) No, the engine was turned off only 3 hours before Lt. Columbo first measured the temperature. Since he spent less than an hour at the house, this was after the robbery.

About Problem 17. (c) 100.

About Problem 19. (c) $\mathrm{UNIV}^{(6)}(0) = 0$, $\mathrm{UNIV}^{(8)}(0) = 42$

About Problem 21. (a) $-2 \le x < 2$ - however, as I write you don't know it converges for $x = -2$, don't worry about this. OTW, you can use the ratio test to see that the radius of convergence is 2 and then see directly that for $x = 2$ it is the harmonic series, which does not converge. Consider end points separately in the rest too. (b) $1 \le x \le 3$. (c) All x. (d) $2/3 \le x \le 4/3$. (e) $x = -5$.