About this handout:

- As always, with handouts, let me know about typos, etc.
- Get started right away. Almost no one can get on top of a thing like this in a day or even two days. Do several a day, like taking daily vitamins, starting right away. Learn what you need to know to be able to do these (not just the solutions), and you will end up in great shape.
- This may be updated as we approach the midterm, but I don’t expect many changes or additions, except for the typos and so on that you will bring to my attention. No updates will occur after 12:01 am, Saturday, May 5.
- The intent is that if you didn’t have to understand it to answer these questions and do these problems with understanding, then you won’t need to know it for the midterm.
- The point above does not mean that it suffices for you to know the answers to these questions and problems. It means that if you understand how to answer these questions and to do these problems, and therefore all problems that require basically the same math, then you will get an A or even an A+ (if you can compute accurately enough).
- Various notations have been used so that you will have to think about the meaning of things. This is intentional. If you cannot figure out the meaning of something, ask. This is another reason to get started right away.
- I won’t provide solutions before the exam. Usually, unless you understand almost everything about the problem and have worked hard on it, seeing a solution won’t help you much. If you find you don’t know how to proceed with some problem, it will help you a lot if you talk to me or the TA - we will help you identify what it is that you need to master to do the problem. It is in the process of trying to solve problems that one learns the most.
- I will provide some answers later, closer to the exam, so that you can check your work. I will do this for problems for which I get several requests. Meantime, if you want to check your answer to a particular problem, email our solution to me and I will get back to you about it.
- Some of these problems are not suitable to put on the time-limited midterm, there is no claim that they are. This is a check list for you, and a muscle building platform.
- Even if this might look scary to you, there is nothing here that is flat out hard. There is a lot that is quite easy.
- This sheet will get incorporated in the corresponding handout for the Final. You are also preparing for the Final while working through this.
Problem 1. Give the precise meaning of the following:

- $D \subseteq \mathbb{R}^2$ is open.
- $D \subseteq \mathbb{R}^3$ is closed.
- $P \subseteq \mathbb{R}^2$ is a boundary point of $D \subseteq \mathbb{R}^2$.
- The open subset $D \subseteq \mathbb{R}^3$ is connected.
- $D \subseteq \mathbb{R}^2$ is a domain.
- $D \subseteq \mathbb{R}^3$ is a region.
- Assuming that $f : \mathbb{R}^2 \to \mathbb{R}$,
  \[
  \lim_{(x,y) \to (7,-1)} f(x,y) = \pi.
  \]
- $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(1,3)$.

In the next two questions, you are to give the definition as the appropriate limit of the appropriate difference quotient.

- Assuming that $f(x_1, x_2, x_3) \in \mathbb{R}$ for $(x_1, x_2, x_3) \in \mathbb{R}^3$,
  \[
  \frac{\partial f}{\partial x_3}(1,5,7).
  \]
- Assuming that $z = f(u, v, s)$ where $f : \mathbb{R}^3 \to \mathbb{R}$,
  \[
  \frac{\partial z}{\partial s}\bigg|_{(1,5,7)}
  \]
- If $w = f(x, y, z)$ where $f : \mathbb{R}^3 \to \mathbb{R}$, the meaning of
  \[
  dw = 7dx - dy + \pi dz \text{ at } (x, y, z) = (1, 2, 3).
  \]
- $z = f(x, y)$ has a total differential at $(x_1, y_1)$.
- $z = f(x, y)$ is differentiable at $(x_1, y_1)$.
- The Jacobian matrix of $f = (f_1, \ldots, f_m)$ where each $f_j : \mathbb{R}^n \to \mathbb{R}$.
- The Jacobian determinant of the $f$ above if $m = n$.
- Assuming that $z = f(u, t, \theta)$ and $p = g(u, t, \theta)$, where $f, g : \mathbb{R}^3 \to \mathbb{R}$,
  \[
  \frac{\partial (p, z)}{\partial (u, \theta)} \text{ and } \frac{\partial (z, p)}{\partial (u, \theta)} \text{ and } \frac{\partial (g, f)}{\partial (t, \theta)}.
  \]
- $|x|$ if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.
- $x \cdot y$ where $x, y \in \mathbb{R}^4$.
- $\cos(\theta)$, where $\theta$ is the angle between nonzero $x, y \in \mathbb{R}^4$.
- $\vec{i}, \vec{j}, \vec{k}$ (you are to express them as lists of numbers).
- $x \times y$ where $x, y \in \mathbb{R}^3$.
- If $x, y \in \mathbb{R}^n$ and $y \neq 0$, define $\text{proj}_y x$, the projection of $x$ along $y$, and $\text{comp}_y x$, the component of $x$ in the direction of $y$.
- A plane in $\mathbb{R}^3$.
- A line in $\mathbb{R}^3$.
- The tangent plane to the level set $\{(x, y, z) : f(x, y, z) = 5\}$ where $f : \mathbb{R}^3 \to \mathbb{R}$.
The gradient of $f: \mathbb{R}^3 \to \mathbb{R}$ and $g: \mathbb{R}^2 \to \mathbb{R}$.

**Problem 2.** Give examples of an open set, a closed set, and a boundary point.

**Problem 3.** Can an open set contain one or more of its boundary points?

**Problem 4.** Must a closed set contain all of its boundary points?

**Problem 5.** Show that if $z = f(x, y)$ is differentiable at $(0, 0)$ and $dz = 7dx - 3dy$ at $(0, 0)$, then
$$\frac{\partial z}{\partial y}(0,0) = -3.$$

**Problem 6.** State the Fundamental Lemma relating the continuity of the partials to the differentiability of $z = f(x, y)$.

**Problem 7.** Compute the following:
$$\frac{\partial^2 z}{\partial u \partial v}, \frac{\partial^2 z}{\partial u^2}, \frac{\partial^2 z}{\partial v^2} \text{ where } z = \ln(u + v^2) + w^3u.$$

**Problem 8.** Compute
$$\frac{\partial^2 z}{\partial u \partial w}|_{(u,v,w)=(\pi,6,7)}.$$

**Problem 9.** Given the following information about $f: \mathbb{R}^4 \to \mathbb{R}$
$$f(3,6,11,-9) = 12, \quad \frac{\partial f}{\partial x_1}(3,6,11,-9) = 3, \quad \frac{\partial f}{\partial x_2}(3,6,11,-9) = -5,$$
$$\frac{\partial f}{\partial x_3}(3,6,11,-9) = 7, \quad \frac{\partial f}{\partial x_4}(3,6,11,-9) = -11,$$
what is your best guess for the value of $f(2.9,6.2,11.1,-8.7)$?

**Problem 10.** Given the following functions:
$$f(x, y, z) = (xy, -\sin(x + z))$$
$$g(x, y) = (x, x, xe^y, x + 3y)$$
$$h(x, y, z) = \cos(xz + y)$$
compute the following

(i) $Df, Dg, Dh, \nabla h$
(ii) $Df(1, \pi, 7), Dg(2, 3), Dh(a, b, c), \nabla h(a, b, c)$. 

In the next two problems, the notation $Df, Dg, Dh, \text{ etc.}$, mean the Jacobian matrices of $f, g, h, \text{ etc.}$
Problem 11. Assume that \( g : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m \) and \( f : V \subset \mathbb{R}^m \to \mathbb{R}^p \), \( x_0 \in U \) and \( y_0 = g(x_0) \in V \). State the chain rule for computing \( Dh(x_0) \) where \( h(x) = f(g(x)) \). Hint: I'll give the answer! The chain rule says that the Jacobian matrix of the composition is the product (in the right order) of the Jacobian of \( f \) and the Jacobian of \( g \) (evaluated at the right places) or \( Dh(x_0) = Df(y_0)Dg(x_0) \).

Problem 12. Suppose \( f : \mathbb{R}^2 \to \mathbb{R}^3 \), \( g : \mathbb{R}^4 \to \mathbb{R}^2 \), \( g(0, 0, 0, 0) = (1, 1) \) and

\[
Dg(0, 0, 0, 0) = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 3 & -3 & 4 & -4 \end{bmatrix}, \quad Df(1, 1) = \begin{bmatrix} \pi & 6 \\ -3 & 5 \\ 7 & 11 \end{bmatrix}.
\]

Suppose \( h(x, y, z, t) = f(g(x, y, z, t)) \). Compute \( \frac{\partial h}{\partial y}(0, 0, 0, 0) \). Note that \( h \) has three components; \( h_2 \) means the second component of \( h \).

**Problem** 13. (One “cos(\( \theta \))” replaced by “sin(\( \theta \)).”) Suppose \( h(\rho, \varphi, \theta) = f(\rho \sin(\varphi) \cos(\theta), \rho \sin(\varphi) \sin(\theta), \rho \cos(\varphi)) \). If we were initially thinking of the real-valued function \( f \) as a function of \( (x, y, z) \) and writing \( f(x, y, z) \), express \( \frac{\partial h}{\partial \rho}, \frac{\partial h}{\partial \varphi}, \frac{\partial h}{\partial \theta} \) in terms of \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \) (say where they get evaluated!) and \( \rho, \varphi, \theta \). If \( f(x, y, z) = xy + z \), evaluate \( \frac{\partial h}{\partial \rho}, \frac{\partial h}{\partial \varphi}, \frac{\partial h}{\partial \theta} \) at \( \rho = \sqrt{2}, \theta = \pi/3, \phi = \pi/4 \). Do the same if all you know is that \( \frac{\partial f}{\partial x}(\sqrt{2}, 1, 2) = -2, \quad \frac{\partial f}{\partial y}(\sqrt{2}, 1, 2) = \pi, \quad \frac{\partial f}{\partial z}(\sqrt{2}, 1, 2) = e. \)

Problem 14. I will not ask you to know by heart the formulas for spherical and cylindrical coords for the midterm. However, this will be needed for the final; you may need to look them up do to this problem, however. I do ask you to know the formulas for polar coordinates for the midterm.

(a) Express the point \((x, y)\) in terms of polar coordinates. Compute the Jacobian matrix and the Jacobian determinant of this transformation.

(b) Express the point \((x, y, z)\) in terms of cylindrical coordinates. Compute the Jacobian matrix and the Jacobian determinant of this transformation.

(c) Express the point \((x, y, z)\) in terms of spherical coordinates. Compute the Jacobian matrix and the Jacobian determinant of this transformation.

Problem 15. Suppose \( z \) is implicitly defined in terms of \((u, v, w)\) by the equation \( \sin(zw) + uz + v^2 = 1 \).

Find \( \frac{\partial z}{\partial v}, \frac{\partial z}{\partial u} \) and \( \frac{\partial^2 z}{\partial u \partial v} \) in terms of \( u, v, w, z \).

Problem 17. Consider the pair of equations

(1) \[ y^3u + x^v = 9, \quad xu + yv = 3. \]

Show they are satisfied by \( y = 2, x = 1, u = 1, v = 1 \). Use the Implicit Function Theorem to show that

(i) There is exactly one pair of continuous functions \( f, g \) defined in an appropriate neighborhood of \((1, 1)\) and satisfying

(2) \[ f(1, 1) = 2, \quad g(1, 1) = 1 \]

such that if \( y = f(u, x) \) and \( v = g(u, x) \), then (1) holds for \((u, x)\) in the appropriate neighborhood. Then compute

\[ \frac{\partial y}{\partial u} \bigg|_{(1,1)}, \quad \frac{\partial y}{\partial x} \bigg|_{(1,1)}, \quad \frac{\partial v}{\partial u} \bigg|_{(1,1)}, \quad \frac{\partial v}{\partial x} \bigg|_{(1,1)}. \]

Hint: If you take differentials, then plug in the values \( y = 2, x = 1, u = 1, v = 1 \), this is quite manageable. Likewise for the variant below.

(ii) There is exactly one pair of continuous functions \( f, g \) defined in an appropriate neighborhood of \((1, 2)\) and satisfying

(3) \[ f(1, 2) = 1, \quad g(1, 2) = 1 \]

such that if \( u = f(v, y) \) and \( x = g(v, y) \), then (1) holds for \((v, y)\) in the appropriate neighborhood. Then compute

\[ \frac{\partial u}{\partial v} \bigg|_{(1,2)}, \quad \frac{\partial u}{\partial y} \bigg|_{(1,2)}, \quad \frac{\partial x}{\partial v} \bigg|_{(1,2)}, \quad \frac{\partial x}{\partial y} \bigg|_{(1,2)}. \]

Problem 18. Compute the following:

(i) \( x \cdot y \) where \( x = (1, 3, -5, 7) \), \( y = (\pi, e, 6, 1) \).

(ii) The cosine of the angle between \( x, y \) above.

(iii) \( (3\vec{i} - 4\vec{k} + \vec{j}) \cdot (3\vec{k} - 4\vec{j} + \vec{i}) \) (read this carefully).

(iv) \( (3, 1, -4) \times (1, -4, 3) \).

(v) \( (3\vec{i} - 4\vec{k} + \vec{j}) \times (3\vec{k} - 4\vec{j} + \vec{i}) \) (read this carefully).

(vi) The direction of \((1, 3, -2)\).

(vii) The direction of \( \vec{i} + 3\vec{j} - 2\vec{k} \).

**Problem** 19. (The only edit was in (vi); (iv) became (v).) Find the equations of the planes \( \mathcal{P} \) in \( \mathbb{R}^3 \) satisfying the given conditions. In each case, give two explicit points
which are in the plane and not among the data given.

(i) \( P \) contains the points \((0, 1, 0), (1, 2, 0), (1, 1, 1)\)

(ii) \( P \) is parallel to the plane \(2x - 3y + 5z + 17 = 0\) and contains the point \((1, 1, 0)\).

(iii) \( P \) is parallel to both of the lines \( t \mapsto t(3, 4, 5) \) and \( t \mapsto (1, 2, 3) + t\vec{i} - 2\vec{j} \)
    and contains the point \((1, 1, 0)\).

(iv) \( P \) is parallel to the line \( t \mapsto t(3, 4, 5) \) and contains the points \((1, 1, 0), (3, 2, 1)\).

(v) \( P \) is perpendicular to the line \( t \mapsto (1, 2, 3) + t\vec{i} - 2\vec{j} \) and contains \((1, -1, 1)\).

(vi) Find the point where the line given in (v) intersects the plane you found.

Problem 20. Let \( A, B \) be \( n \)-vectors with \( B \neq 0 \). Show that there is exactly one vector \( C \) which is orthogonal to \( B \) and one number \( \alpha \) such that \( A = \alpha B + C \). Give formulas for \( \alpha \) and \( C \) and verify that your formulas do the job. Hint: Take the inner-product of both sides with \( B \) and solve the result for \( \alpha \), expressing it in terms of \( A, B \).

Draw a picture. Compute the projection of \( 3\vec{i} - 4\vec{j} + 5\vec{k} \) along \((1, 2, -1)\) and the component of \( 3\vec{i} - 4\vec{j} + 5\vec{k} \) in the direction of \((1, 2, -1)\) as well.

Problem 21. Let \( N \in \mathbb{R}^3, N \neq 0 \). Consider the plane \( P \) whose equation is \( N \cdot (x - x_0, y - y_0, z - z_0) = 0 \).

If \( P \in \mathbb{R}^3 \), find the one and only point \( Q \in P \) such that \( P - Q \) is a multiple of \( N \), that is, \( P - Q \) is also a normal to the plane. Hint: Write \( P = \alpha N + Q \) where this time, in contrast to Problem 20, \( Q \cdot N = N \cdot (x_0, y_0, z_0) \) (this puts \( Q \) in the plane) and use the hint in Problem 20. Show that the \( Q \) you compute this way is in fact in \( P \). Then show that

\[
|P - Q| = \frac{|(P - (x_0, y_0, z_0)) \cdot N|}{|N|}.
\]

If \( P \) is given by \( Ax + By + Cz + D = 0 \) and \( P = (x_1, y_1, z_1) \), show that (4) just says that

\[
|P - Q| = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.
\]

Draw a picture which shows why either (4) or (5) give the distance from \( P \) to \( P \). Calculate the distance from \((1, 2, -3)\) to the plane \( 6x + 8z = -7y + 7 \).

Problem 22. Compute the distance of the point \((3, 2, 1)\) from the line \( x = 1 + 3t, y = -1 + 5t, z = -6t \).

Problem 23. Define the gradient \( \nabla f \) (aka “grad \( f \)”) of a function \( f : \mathbb{R}^3 \to \mathbb{R} \). Define the derivative of \( f \) along a vector \( v \) at a point \((x_0, y_0, z_0)\). Explain the difference between the directional derivative of \( f \) along \( v \) and the directional derivative of \( f \) in the direction of \( v \) (if \( v \neq 0 \)).
Problem 24. Compute the directional derivative of $f(x, y, z) = e^{xy} \cos(z) + z$ along $v$ and in the direction of $v$ for $v = 3\hat{i} - 4\hat{k}$, and any other $v$ you make up.

The text introduces the terminology “level curves” for functions defined on subsets of $\mathbb{R}^2$ and “level surfaces” for functions defined on subsets of $\mathbb{R}^3$ in Section 2.3. In general, one talks about “level sets” in any number of dimensions. If $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$, the “$L$-level set of $f$” is

$\{(x_1, \ldots, x_n) \in D : f(x_1, \ldots, x_n) = L\}$.

Eg, the 27-level set of $f(x, y, z) = x^2 + y^2 + z^2$ is the set of points whose coordinates satisfy $x^2 + y^2 + z^2 = 27$.

If we don’t care to name the level $L$, we just say “a level set.”

Problem 25. Show that if $S$ is a level set of $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $c(t)$ is a path in the level set, that is, $c(t) \in S$ for all $t$, then $\nabla f(c(t)) \cdot c'(t) = 0$. Explain why we interpret this as saying that $\nabla f(x_0, y_0, z_0)$ is perpendicular to the $f(x_0, y_0, z_0)$-level set of $f$.

Problem 26. Find the equation of the tangent plane to the graph of $f(x, y) = x^2 + 11xy$ at the point $(1, 1, 12)$ on its graph.

Problem 27. Give the equation of the tangent plane to the surface $e^{zx} + x^3y^3 + z + 2y = 6$ at the point $(0, 0, 5)$.

Problem 28. Three ducks, Duck 1, Duck 2 and Mildred are swimming in a Wisconsin lake in November. That water is cold! The ducks charge about seeking warmer water. In this wonder world, all three ducks are at the same location, $(0,0)$ (I’d like to see that!). Duck 1 is headed in the direction of $(1,1)$ with speed 4ft/sec, Duck 2 is headed in the direction of $(1,-5)$ with speed 5ft/sec. The temperature of the water is given by $T(x, y) = 33 + 4x + y + xy \degree$, where $x, y$ are in feet. Which of Duck 1 and Duck 2 is warming up faster? Mildred is traveling with speed 2. What direction should she head in to warm up as fast as possible? If she heads in this direction, does she warm up faster than the speedier Ducks 1 and 2 or not?

Problem 29. Suppose $V(x, y, z) = -1/r$ where $r = \sqrt{x^2 + y^2 + z^2}$. Show that

$\nabla V = \frac{1}{r^2} \hat{r}$ where $\hat{r} = (x, y, z)$.

Note that $\hat{r}/r$ is the direction of $\hat{r}$.

Problem 30. State the Cauchy - Schwarz inequality.

Problem 31. This problem won’t be on the midterm. I put it here as it is somewhat likely to be on the final. Let $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ be 4-vectors. The Cauchy - Schwarz inequality states that

$|x \cdot y| \leq |x||y|$.
Prove that this is true as was done in class. Do NOT use $x \cdot y = |x||y|\cos(\theta)$, which stands as the definition of the angle between $x$ and $y$, a definition which is justified by the CS inequality.

**Problem** 32. (added 5/3) Let $w = f(u, v)$. Suppose that $u = g(t), v = h(t)$, and

$$g(0) = 3, g'(0) = 1, g''(0) = -2, h(0) = 4, h'(0) = \pi, h''(0) = 0.$$ 

Suppose also that

$$\frac{\partial f}{\partial u} = 4u^3 + v^2, \quad \frac{\partial f}{\partial v} = 2uv.$$ 

Compute

$$\left.\frac{d^2w}{dt^2}\right|_{t=0}.$$ 

**Problem** 33. (added 8:52 pm, 5/5) Consider the change of variables (or coordinates) $x = u^2 - v^2, y = uv$ which expresses the coordinates $x, y$ in terms of the coordinates $u, v$. Find the coefficients $a(u, v), b(u, v), c(u, v), d(u, v)$ for which

$$\frac{\partial}{\partial x} = a(u, v)\frac{\partial}{\partial u} + b(u, v)\frac{\partial}{\partial v}, \quad \frac{\partial}{\partial y} = c(u, v)\frac{\partial}{\partial u} + d(u, v)\frac{\partial}{\partial v}.$$ 

Then express

$$\frac{\partial^2 w}{\partial x \partial y}$$ 

in the $u, v$ coordinates.