Supplement to Linköping lecture: version 10/9/04

SAMPLE DERIVATIONS

The material here is a product of the community, including Bhattacharya, Evans, Gariepy, Jensen, Lindqvist, Manfredi, the Tour team, (Aronsson, C. and Juutinen) and others. But others are not to blame for the probably numerous errors. It may be that a couple of things here are clearer than elsewhere and a couple of remarks are new. These are only samples, not the most significant and difficult points, such as the uniqueness and the regularity beyond Lipschitz. The point is to provide easy access to some sample basic arguments, without burdensome details, notation, definitions and preliminaries.

#1: AM for \( F = \text{Lip} \iff \text{COMPARISON WITH CONES} \)

Of course, “AM” is short for absolutely minimizing. First, let \( u \in C(U) \) enjoy comparison with cones in \( U \), ie,

\[
\begin{align*}
\text{(a)} \quad & \min_{w \in \partial V} (u(w) - a|w - z|) \leq u(x) - a|x - z| \quad \text{and} \\
\text{(b)} \quad & u(x) - a|x - z| \leq \max_{w \in \partial V} (u(w) - a|w - z|). \\
\end{align*}
\]

for \( V \subset U, x \in V, z \notin V \) and \( a \in \mathbb{R} \). Note that the condition (0.1) (a) - which is comparison with cones from above - is equivalent to the condition (0.2) (a) below. Likewise, comparison with cones from below ((0.1) (b)) may be restated as (0.2) (b).

\[
\begin{align*}
\text{(a)} \quad & C + a|x - z| \leq u(x) \quad \text{holds for all } x \in V \text{ whenever it holds for } x \in \partial V \\
\text{(b)} \quad & u(x) \leq C + a|x - z| \quad \text{holds for all } x \in V \text{ whenever it holds for } x \in \partial V. \\
\end{align*}
\]

We claim that then, for any \( x \in V \),

\[
\text{Lip}(u, \partial(V \setminus \{x\})) = \text{Lip}(u, \partial V). 
\]

To see this we need only check that for \( v \in \partial V, x \in V \),

\[
\text{u(v) - Lip}(u, \partial V)|x - v| \leq u(x) \leq u(v) + \text{Lip}(u, \partial V)|x - v|. 
\]

As each of the above inequalities holds for \( x \in \partial V \) and \( u \) enjoys comparison with cones, the inequalities indeed hold if \( x \in V \). Let \( x, y \in V \). Using (0.3) twice,

\[
\text{Lip}(u, \partial V) = \text{Lip}(u, \partial(V \setminus \{x\})) = \text{Lip}(u, \partial(V \setminus \{x, y\})). 
\]

Since \( x, y \in \partial(V \setminus \{x, y\}) \), we have \( |u(x) - u(y)| \leq \text{Lip}(u, \partial V)|x - y| \), and hence \( u \) is absolutely minimizing for \( F = \text{Lip} \).
Suppose now that $u$ is absolutely minimizing for $F = \text{Lip}$. Assume that $V \subset U$, $z \notin V$ and set

\begin{equation}
W = \{ x \in V : u(x) - a|x - z| > \max_{w \in \partial V} (u(w) - a|w - z|) \}.
\end{equation}

We want to show that $W$ is empty. If it is not empty, then it is open and

\begin{equation}
u(x) = a|x - z| + \max_{w \in \partial V} (u(w) - a|w - z|) =: C(x) \text{ for } x \in \partial W.
\end{equation}

Therefore $u = C$ on $\partial W$ and $\text{Lip}(u, W) = \text{Lip}(C, \partial W) = |a|$ since $u$ is absolutely minimizing. Now if $x^0 \in W$, the line $t \mapsto x^0 + t(x^0 - z)$ contains a segment in $W$ containing $x^0$ which meets $\partial W$ at its endpoints. Since $t \mapsto C(x^0 + t(x^0 - z)) - C(x^0) = at|x^0 - z|$ is linear on this segment, with slope $a|x^0 - z|$, while $t \mapsto u(x^0 + t(x^0 - z)) - C(x^0)$ has $|a||x^0 - z|$ as a Lipschitz constant and the same values at the endpoints; therefore it is the same function. Thus

\begin{equation}u(x^0 + t(x^0 - z)) - C(x^0) = C(x^0 + t(x^0 - z)) - C(x^0)
\end{equation}
on the segment, whence $u(x^0) = C(x^0)$, a contradiction to $x^0 \in W$. Thus $W$ is empty.

**#2:** AM for $F = F_\infty \implies$ COMPARISON WITH CONES $\implies$ AM for $F = \text{Lip}$.

The proof just above works, because $|DC| = |a|$ in $W$ and $u$ AM for $F = F_\infty$ implies, by its definition, that $u$ is locally Lipschitz continuous and $|Du| \leq |a|$ almost everywhere in $W$. Hence $|a|$ is a Lipschitz constant for $u$ on line segments in $W$. This yields the first arrow above, while the second arrow was established in **#1**.

**#3** MISCELLANY: WARMING UP

Let $u : U \to \mathbb{R}$ be upper-semicontinuous (just for a minor technical tap-dance :) and satisfy comparison with cones from above. Using the form (0.2) (a) of this condition, we find

\begin{equation}
u(x) \leq u(y) + \max_{|w - y| = r} \left( \frac{u(w) - u(y)}{r} \right) |x - y|
\end{equation}
for $x \in B_r(y) \subset U$. The inequality (0.7) holds as asserted because it trivially holds for $x \in \partial (B_r(y) \setminus \{y\})$.

Let us weaken this by replacing the max over the sphere with the max over the ball:

\begin{equation}
u(x) \leq u(y) + \max_{|w - y| \leq r} \left( \frac{u(w) - u(y)}{r} \right) |x - y|.
\end{equation}
Assume now that $|x - y| < r$ and $u(x) = \max_{B_r(x)} u$. Then we may replace $u(w)$ by $u(x)$ above to conclude that $u(x)(1 - |x - y|/r) \leq u(y)(1 - |x - y|/r)$, which implies that $u(x) \leq u(y)$. Since also $u(x) \geq u(y)$, we conclude that $u(x) = u(y)$. Since this is true for all $y$ such that $x \in B_r(y) \subset U$ and $u(x) = \max_{B_r(x)} u$, it is true if $B_R(x) \subset U$, $u(x) = \max_{B_R(x)} u$ and $|y - x| < R/2$. Thus if $u$ has a local maximum point, it is constant in a ball around that point. This guarantees that the maximum of $u$ over any closed ball is attained in the boundary. There is no difference between (0.8) and (0.7).

A variant of (0.7) is

\[
\begin{align*}
(0.9) \quad u(x) & \leq \left( \max_{|w - y| \leq r} u(w) \right) \frac{|x - y| - s}{r - s} + \left( \max_{|w - y| \leq s} u(w) \right) \frac{r - |x - y|}{r - s},
\end{align*}
\]

for $0 \leq s \leq |x - y| \leq r$. The inequality is obvious for $|x - y| = s, r$, so it holds as asserted. Taking the maximum of $u(x)$ over $B_r(x)$, where $s \leq \tau \leq r$ yields

\[
(0.10) \quad \max_{|w - y| \leq \tau} u(w) \leq \left( \max_{|w - y| \leq r} u(w) \right) \frac{\tau - s}{r - s} + \left( \max_{|w - y| \leq s} u(w) \right) \frac{r - \tau}{r - s},
\]

which says that $\tau \mapsto \max_{|w - y| \leq \tau} u(w)$ is convex.

#3 COMPARISON WITH CONES IMPLIES GRADIENT ESTIMATE AND HARNACK

First we show that if an upper-semicontinuous $u$ satisfies (0.7), then it is locally Lipschitz continuous. To start, we assume that $u$ in (0.7) satisfies $u \leq 0$, so that the maximum on the right can be dropped. Thus we have, written three equivalent ways,

\[
\begin{align*}
(a) \quad u(x) & \leq \left( 1 - \frac{|x - y|}{r} \right) u(y); \\
(b) \quad -u(y) & \leq -u(x) \left( \frac{r}{r - |x - y|} \right); \\
(c) \quad u(x) - u(y) & \leq -\frac{|x - y|}{r} u(y).
\end{align*}
\]

Either of (0.11) (a) or (b) is a Harnack inequality; (b) is displayed just because you might prefer the nonnegative function $-u$, which is lower-semicontinuous and enjoys comparison with cones from below, to $u$. If $u(x) \neq 0$, either estimates the ratio $u(y)/u(x)$ by quantities not depending on $u$. Taking the limit inferior as $y \to x$ on the right of (0.11)(a), we find that $u$ is lower-semicontinuous as well as upper-semicontinuous, so it is continuous. If also $B_r(x) \subset U$, we may interchange $x$ and $y$ in (0.11) (c) and conclude, from the two
relations, that
\[ |u(x) - u(y)| \leq - \min(u(x), u(y)) \frac{|x - y|}{r - |x - y|}. \]

As \( u \) is locally bounded, being continuous, we conclude that it is also locally Lipschitz continuous. Therefore \( u \) is differentiable a.e. by Rademacher’s theorem. If \( u \leq 0 \) does not hold, the above applies to, say, \( u(x) - \max_{B_{2R}(z)} u \) instead, provided that we keep \( x, y \in B_R(z), B_{2R}(z) \subset U \), and \( r < R \) for some \( z \in U, R > 0 \). The conclusion is that an upper-semicontinuous function satisfying comparison with cones from above is locally Lipschitz continuous. One doesn’t need to pass through a “Harnack inequality” here, it is just one way to do some bookkeeping.

Next we estimate the gradient directly from (0.7), no longer assuming that \( u \) has a sign. We have
\[ \frac{u(x) - u(y)}{|x - y|} \leq \max_{|w - y| = r} \left( \frac{u(w) - u(y)}{r} \right). \]

Assuming that \( y \) is a point of differentiability of \( u \) and \( Du(y) \neq 0 \), put \( x = y + \lambda Du(y) \) with \( \lambda > 0 \) in this relation to find
\[ \frac{u(x) - u(y)}{|x - y|} = \lambda \langle Du(y), Du(y) \rangle + o(\lambda) \leq \max_{|w - y| = r} \left( \frac{u(w) - u(y)}{r} \right) \]
and let \( \lambda \downarrow 0 \). The result is
\begin{equation}
(0.12) \quad |Du(y)| \leq \max_{|w - y| = r} \left( \frac{u(w) - u(y)}{r} \right) \quad \text{for} \ r < \text{dist} (y, \partial U).
\end{equation}

We will work with (0.12) as if \( Du \) were everywhere defined and continuous, which it is not. However, our conclusions are all easily justified (an exercise for you).

No information has been lost in passing to (0.12). It can be integrated to recover (0.8). Suppose that \( t \to \gamma(t) \) is a \( C^1 \) curve in \( U \). Using (0.12) with \( y = \gamma(t) \),
\begin{equation}
(0.13) \quad \left| \frac{d}{dt} u(\gamma(t)) \right| = |\langle Du(\gamma(t)), \dot{\gamma}(t) \rangle| \leq |Du(\gamma(t))| |\dot{\gamma}(t)|
\end{equation}
\[ \leq \max_{|w - \gamma(t)| = r} \left( \frac{u(w) - u(\gamma(t))}{r} \right) |\dot{\gamma}(t)| \quad \text{for} \ r < \text{dist} (\gamma(t), \partial U). \]
It is convenient to rewrite this as
\begin{equation}
(0.14) \quad \pm \frac{d}{dt} u(\gamma(t)) + \frac{|\dot{\gamma}(t)|}{r} u(\gamma(t)) \leq \left( \max_{|w - \gamma(t)| = r} u(w) \right) \frac{|\dot{\gamma}(t)|}{r} \quad \text{for} \ r < \text{dist} (\gamma(t), \partial U).
\end{equation}
Assuming now that \( x \in B_r(y) \subset U \), put \( \gamma(t) = y + t(x-y), 0 \leq t \leq 1 \). Then \( \text{dist}(\gamma(t), \partial U) \geq r - |y - \gamma(t)| = r - t|x-y| \). So we may replace \( r \) in (0.14) by \( r - t|x-y| \).

Using also \( B_{r-t|x-y|}(\gamma(t)) \subset B_r(y) \), we end up with

\[
\frac{d}{dt} u(\gamma(t)) + \frac{|x-y|}{r-t|x-y|} u(\gamma(t)) \leq \left( \max_{|w-y| \leq r} u(w) \right) \frac{|x-y|}{r-t|x-y|}.
\]

This simple differential inequality integrated over \( 0 \leq t \leq 1 \) yields (0.8).

If instead we assume that \( x, y \in B_r(z) \subset B_R(z) \subset U \), then \( \text{dist}(\gamma(t), \partial U) \geq R - r \).

In the event \( u \leq 0 \), we deduce from (0.15) that

\[
\pm \frac{d}{dt} u(\gamma(t)) + \frac{|x-y|}{R-r} u(\gamma(t)) \leq 0,
\]

which (with the plus sign) integrates to

\[
e^{\frac{|x-y|}{R-r}} u(x) \leq u(y).
\]

Replacing \( u \) by \( v = -u, v \geq 0 \) enjoys comparison with cones from below and the estimate is

\[
v(y) \leq e^{\frac{|x-y|}{R-r}} v(x).
\]

For another example, assume that \( U = \{(x_1, \ldots, x_n) : x_n > 0\} \) is a half-space. Then, by (0.12) and \( \text{dist}(x, \partial U) = x_n \), if \( u \leq 0 \),

\[
\pm \frac{\partial u(x)}{\partial x_n} \leq |Du(x)| \leq \frac{-u(x)}{x_n}.
\]

or

\[
\pm \frac{\partial u(x)}{\partial x_n} + \frac{u(x)}{x_n} \leq 0.
\]

With the plus sign, this says that \( x_n \mapsto x_n u(x_1, \ldots, x_n) \) is nonincreasing and, with the minus sign, it says that \( x_n \mapsto u(x_1, \ldots, x_n)/x_n \) is nondecreasing.

The “increasing/decreasing” statements above are found in the paper On the behavior of \( \infty \)-harmonic functions in some special unbounded domains by T. Bhattacharya. See this paper for other references and discussion. As we have just seen, “Harnack” type estimates which follow from a one sided assumption, eg, comparison with cones from above or from below, are rather simple. This is no longer the case, for example, with Bhattacharya’s boundary Harnack inequality for non-negative infinity harmonic functions in a half-space, which uses “both sides”.
#4 COMPARISON WITH CONES \implies INFINITY HARMONIC

You may prefer other proofs, but this one is direct; it does not use contradiction. First we use comparison with cones from above, (0.7) above, which we rewrite as

\begin{equation}
\tag{0.19}
u(x) - u(y) \leq \max_{|w-y|=r} (u(w) - u(x)) \frac{|x-y|}{r - |x-y|}
\end{equation}

for \(x \in B_r(y) \subseteq U\), to show that if \(u\) is twice differentiable at \(x\), namely, there is a \(p \in \mathbb{R}^n\) and a symmetric \(n \times n\) matrix \(X\) such that

\begin{equation}
\tag{0.20}
u(z) = u(x) + \langle p, z - x \rangle + \frac{1}{2} \langle X(z - x), z - x \rangle + o \left( |z - x|^2 \right),
\end{equation}

in which case we put \(p = Du(x), X = D^2u(x)\),

\begin{equation}
\tag{0.21}\Delta_\infty u(x) = \langle D^2u(x)Du(x), Du(x) \rangle = \langle Xp, p \rangle \geq 0.
\end{equation}

That is, comparison with cones from above implies \(\Delta_\infty u \geq 0\) at points of twice differentiability.

We are going to plug (0.20) into (0.19) with two choices of \(z\). First, on the left of (0.19), we choose \(z = y = x - \lambda p\) where \(p\) is from (0.20), and expand \(u(x) - u(y)\) according to (0.20). Next, let \(w_{r,\lambda}\) be a value of \(w\) for which the maximum on the right of (0.19) is attained and expand \(u(w_{r,\lambda}) - u(x)\) according to (0.20). This yields, after dividing by \(\lambda > 0\),

\begin{equation}
\tag{0.22}|p|^2 + \lambda^2 \langle Xp, p \rangle + o(\lambda) \leq \left( \langle p, w_{r,\lambda} - x \rangle + \frac{1}{2} \langle X(w_{r,\lambda} - x), w_{r,\lambda} - x \rangle \right) \frac{|p|}{r - |x|} + o((r + \lambda)^2)/r.
\end{equation}

Sending \(\lambda \downarrow 0\) yields

\begin{equation}
\tag{0.23}|p|^2 \leq \left( \langle p, \frac{w_r - x}{r} \rangle + \frac{1}{2} \langle X \left( \frac{w_r - x}{r} \right), w_r - x \rangle \right) |p| + o(r)
\end{equation}

\begin{align*}
&\leq |p|^2 + \frac{1}{2} \langle X \left( \frac{w_r - x}{r} \right), w_r - x \rangle |p| + o(r),
\end{align*}

where \(w_r\) is any limit point of the \(w_{r,\lambda}\) as \(\lambda \downarrow 0\) and therefore \(w_r \in \partial B_r(x)\) - so \((w_r - x)/r\) is a unit vector. Since the second term inside the parentheses on the right of the first inequality above has size \(r\) and \((w_r - x)/r\) is a unit vector, it follows from the first inequality that \((w_r - x)/r \to p/|p|\) as \(r \downarrow 0\). (We are assuming that \(p \neq 0\), as we
may.) Then the inequality of the extremes in (0.23), after dividing by \( r \) and letting \( r \downarrow 0 \), yields \( 0 \leq \langle Xp, p \rangle \), as desired.

This set up is a bit more awkward than is necessary for the result obtained so far. It is set up this way to make the next remark easy. If \( x \) is a local maximum point of \( u - \phi \) for some smooth \( \phi \), then

\[
\phi(x) - \phi(y) \leq u(x) - u(y) \quad \text{and} \quad u(w) - u(x) \leq \phi(w) - \phi(x)
\]

for \( y, w \) near \( x \). That is, we may replace \( u \) by \( \phi \) in (0.19). By what was just shown, it follows that \( \Delta_{\infty} \phi(x) \geq 0 \). That is, \( u \) is a viscosity solution of \( \Delta_{\infty} u \geq 0 \) if it satisfies comparison with cones from above.

#5 INFINITY HARMONIC IMPLIES COMPARISON WITH CONES

Suppose that \( \Delta_{\infty} u \geq 0 \) on the bounded set \( U \). One computes

\[
\Delta_{\infty} G(|x|) = G''(|x|)G'(|x|)^2
\]

if \( x \neq 0 \) and from this that

\[
\Delta_{\infty}(a|x - z| - \gamma|x - z|^2) = -2\gamma(a - 2\gamma|x - z|)^2 < 0
\]

for all \( x \in U, x \neq z \), if \( \gamma > 0 \) is small enough. But then if \( \Delta_{\infty} u \geq 0 \) (in the viscosity sense), \( u(x) - (a|x - z| - \gamma|x - z|^2) \) cannot have a local maximum in \( U \) different from \( z \), by the very definition of a viscosity solution of \( \Delta_{\infty} u \geq 0 \). Thus if \( z \not\in V \subset U \) and \( x \in V \), we have

\[
u(x) - (a|x - z| - \gamma|x - z|^2) \leq \max_{w \in \partial V} (u(w) - (a|w - z| - \gamma|w - z|^2)).
\]

Now let \( \gamma \downarrow 0 \).

#6 AM FOR \( F = \text{Lip} \implies \) AM for \( F = F_{\infty} \).

This is the “missing link”. We showed AM for \( F = \text{Lip} \iff \) comparison with cones \iff \) infinity harmonic. We also showed that AM for \( F = F_{\infty} \implies \) comparison with cones. We need to know \textit{something} implies AM for \( F = F_{\infty} \). I don’t know a really simple proof, but there are papers I haven’t absorbed yet, including one by Y. Yu and another by Champion and DePascale. We’ll all be smarter by the end of the meeting.

Jensen did it this way. Suppose we know the existence of and uniqueness of a function \( u \in C(\overline{U}) \) such that \( \Delta_{\infty} u = 0 \) in \( U \) and \( u = b \) on \( \partial U \) whenever \( U \) is open and bounded and that this function is AM for \( F = F_{\infty} \) (Jensen). Suppose \( u \) is AM in \( U \) for \( F = \text{Lip} \).
Then $\Delta_\infty u = 0$ by the above. This does it, since $u$ is then also the one and only function which is AM for $F = F_\infty$ in $V \subset\subset U$ which agrees with $u$ on $\partial V$. But the argument requires “everything”: harder existence and uniqueness. We say “harder existence”, since existence for AM for $F = \text{Lip}$ is pretty easy (see Tour).

Alternatively, see Tour for a generalized CEG proof (what language is this!?) which does not require uniqueness or existence, but directly shows comparison with cones implies AM for $F = F_\infty$. This proof works in a generality that goes beyond our current ability to prove uniqueness.