Set up: $A$ is hyperbolic, $\mathbb{R}^n = E_s \oplus E_u$ is the decomposition of $\mathbb{R}^n$ into the $A$ invariant spaces such that $A|_E$s has eigenvalues of negative real part and $A|_{E_u}$ has eigenvalues of positive real part. $P_s, P_u$ are the corresponding projections on $E_s, E_u$ respectively. The time $t$ map for the ivp

$$
\dot{x} = Ax + f(x)
$$

is $\psi_t$; for any solution of (1.1) $x(t) = \psi_t(x(0))$. Here $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies

$$
\|f(x)\| \leq C \quad \text{and} \quad \|f(x) - f(y)\| \leq L\|x - y\| \quad \text{and} \quad f(0) = 0.
$$

**Theorem 1.1.** Let (1.2) hold. Then there is a number $0 < L_0(A)$ such that if $L < L_0(A)$ then there is unique bounded and continuous $p : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$
e^{tA}x + p(e^{tA}x) = \psi_t(x + p(x)) \quad \text{for} \quad x \in \mathbb{R}^n, t \in \mathbb{R}.
$$

Moreover, $x \rightarrow x + p(x)$ is a homeomorphism of $\mathbb{R}^n$.

To begin, let us first note that if $p$ is bounded, then (1.3) guarantees that $I + p$ is a homeomorphism. Indeed, any continuous bounded perturbation of the identity is onto $\mathbb{R}^n$, so we only need to show that $x + p(x) = y + p(y)$ implies $x = y$. However, from (1.3) we find then that $e^{tA}(x - y) = -p(e^{tA}x) + p(e^{tA}y)$ and the right-hand side is bounded independently of $t$. Since $A$ is hyperbolic, this forces $x = y$.

It remains to establish the existence and uniqueness of $p$. We need the standard observation:

**Proposition 1.2.** Let $A$ be hyperbolic and $g : \mathbb{R} \rightarrow \mathbb{R}^n$ be continuous and bounded. Then a solution $x(t)$ of

$$
\dot{x} = Ax + g(t)
$$
on $\mathbb{R}$ is bounded for $0 \leq t$ (respectively, $t \leq 0$) if and only if

$$
P_u x(0) + \int_0^\infty e^{-\tau A}P_u g(\tau) d\tau = 0
$$
(respectively $P_s x(0) - \int_\infty^0 e^{-\tau A}P_s g(\tau) d\tau = 0$).
To continue the proof of Theorem 1.1 we differentiate (1.3) to find
\[ Ae^{tA}x + \frac{d}{dt}p(e^{tA}x) = A\psi_t(x + p(x)) + f(\psi_t(x + p(x))) \]
\[ = A(e^{tA}x + p(e^{tA}x)) + f(e^{tA}x + p(e^{tA}x)) \]
so
\[
\frac{d}{dt}p(e^{tA}x) = Ap(e^{tA}x) + f(e^{tA}x + p(e^{tA}x))
\]
(1.5)
Clearly the argument is reversible, and (1.5) implies (1.3).

Since \( f \) and \( p \) in (1.5) are bounded, we conclude from Proposition 1.2 that necessarily
\[
Pu p(x) + \int_0^\infty e^{-\tau A}Pu f(e^{\tau A}x + p(e^{\tau A}x)) d\tau = 0,
\]
(1.6)
\[
Psp p(x) - \int_{-\infty}^0 e^{-\tau A}Ps f(e^{\tau A}x + p(e^{\tau A}x)) d\tau = 0.
\]
This fixed point problem for \( p \) trivially submits to the contraction mapping theorem if the Lipschitz constant for \( f \) is sufficiently small (depending on \( A \)). There is a unique bounded and continuous fixed point.

Assuming (1.6) holds, we deduce the integral form of (1.5), namely
\[
p(e^{tA}x) = e^{tA} \left( p(x) + \int_0^t e^{-\tau A} f(e^{\tau A}x + p(e^{\tau A}x)) d\tau \right).
\]
This arises by replacing \( x \) by \( e^{tA}x \) in each of the relations in (1.6) and adding.

Happy New Year!