UNIQUENESS OF ∞-HARMONIC FUNCTIONS AND THE EIKONAL EQUATION

MICHAEL G. CRANDALL, GUNNAR GUNNARSSON, & PEIYONG WANG

Abstract. Comparison results are obtained between infinity subharmonic and infinity superharmonic functions defined on unbounded domains. The primary new tool employed is an approximation of infinity subharmonic functions that allows one to assume that gradients are bounded away from zero. This approximation also demystifies the proof in the case of a bounded domain. A second, quite different, topic is also taken up. This is the uniqueness of absolutely minimizing functions with respect to other norms besides the Euclidean, norms that correspond to comparison results for partial differential equations which are quite discontinuous.

Introduction

In this paper, we take up two topics related to the uniqueness of solutions of the Dirichlet problem for the ∞ - Laplace equation $\Delta_\infty u = 0$, where the "∞-Laplacian" Δ_∞ is given on smooth functions by

$$\Delta_\infty u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j}.$$

The first topic is the study of the uniqueness of solutions of Dirichlet problems for the ∞-Laplace equation in unbounded domains, for example, exterior domains. To explain, let U be an open subset of $\mathbb{R}^n$ and assume that

$$\text{(H1)} \qquad u, v \in C(\overline{U}) \text{ and } \Delta_\infty u \geq 0 \text{ and } \Delta_\infty v \leq 0 \text{ in } U.$$

Can we conclude that

$$\text{(0.1)} \qquad u(x) - v(x) \leq \sup_{\partial U}(u - v) \text{ for } \ x \in U?$$

The inequalities in (H1) have to be interpreted suitably, see below. R. Jensen [11] showed that the answer is "yes" if U is bounded, and Barles and Busca [2] gave another, quite different, proof in this case. We are interested in cases in which U is unbounded. If we allow u, v to grow linearly, the answer in general is no. The simplest example of this is $U = \mathbb{R}^n \setminus \{0\}$. Then linear functions $u = \langle p, x\rangle, v(x) = \langle q, x\rangle$ with $p \neq q$ satisfy (H1), but not (0.1). However, if the growth of u and v is sublinear, one expects that the answer to be "yes," and it is if $U = \mathbb{R}^n \setminus \{0\}$. In this particular case, this is easy to show. However, it was unknown until the current work whether or not this remains true even if $U = \mathbb{R}^n \setminus \{w, z\}$ is the complement of two points.

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In Section 3 we show, more generally, that if

$$\partial U \text{ is compact} \tag{0.2}$$

and

$$\limsup_{|x|\to\infty} \frac{u(x)}{|x|} \le 0 \text{ and } \liminf_{|x|\to\infty} \frac{v(x)}{|x|} \ge 0, \tag{0.3}$$

then the answer to (0.1) is indeed "yes." Cases in which (0.2) holds and U is unbounded, we refer to as *exterior problems*. We reduce exterior problems to the bounded domain case via a striking approximation of ∞-subharmonic functions (i.e., functions u such that $\Delta_\infty u \ge 0$). This uniqueness result complements an existence result of G. Aronsson, M. Crandall and P. Juutinen [1], Section 3, which provides solutions of the Dirichlet problem for $\Delta_\infty u = 0$ for any U, provided that the Dirichlet data grow at most linearly. If the data are bounded, as they are in the case above, then this existence result provides solutions obeying the same bounds.

At the heart of our proofs is an approximation which replaces u on the set $\{x \in U : |Du(x)| < \varepsilon\}$ by a solution of the eikonal equation $\varepsilon - |Dw| = 0$. The result is a new infinity subharmonic function u_ε which satisfies $|Du_\varepsilon(x)| \ge \varepsilon$ in U. As the previous proofs dealt with zeros of the gradient - a major difficulty in the uniqueness theory - in subtle or complex ways, this approximation clarifies the situation substantially. In spirit, our proofs are much closer to the ideas of Jensen [11] than to those of Barles and Busca [2]. In fact, Jensen used the obstacle problem for the gradient associated with the operator formally given by $\Delta_\infty^\varepsilon v := \min(\Delta_\infty v, |Dv| - \varepsilon)$ in his proof. If u is ∞-harmonic, then Barron and Jensen [3], showed that $\Delta_\infty^\varepsilon u_\varepsilon = 0$, as was brought to our attention by Jensen. In particular, they showed that u_ε is ∞-subharmonic. However, they did not go on to obtain the approximation properties of this procedure, as applied to merely ∞-subharmonic functions, and go on to prove comparison results, as we will do.

While we wrote $|Du(x)|$, etc, above, the functions being dealt with need not be differentiable at every point x. The exact definition of the set on which u is modified involves instead the "local Lipschitz constant," introduced in [1], as explained in Section 1. In a related spirit, the expressions $\Delta_\infty u \ge 0$, etc, are all understood in the viscosity sense. See, e.g., M. Crandall, H. Ishii and P. L. Lions [8]; alternatively, there is sufficient explanation in [1] or in [6], which will serve as our references for a variety of basic facts, and in Section 1. However, in the end, the proofs of this paper will use basic results from the theory of viscosity solutions.

In Section 3 we first prove comparison in the event that U is bounded. As mentioned, this was previously established in [11] and [2], with two different proofs. Given the approximation result, the proof is immediate via the standard technology of viscosity solutions. Then we use the approximation again to quickly reduce exterior problems to the previous case.

The case of unbounded ∂U is taken up in Section 4. Our results here are not definitive; we assume that u, v are bounded and uniformly continuous (at least near ∂U).

The equation $\Delta_\infty u = 0$ is one way, among many, to characterize "absolutely minimizing functions," in the case of the Euclidean norm, as defined in the introduction to [1] and reviewed in Section 1 below. If we use another norm to determine the notion absolutely minimizing functions, the associated partial differential equation changes in response. For example, if the maximum norm $|x|_\infty = \max_j |x|_j$ is used instead, the inequalities of (H1) become

$$\max_{q\in \operatorname{sign}(Du)} \big\langle (D^2u)q, q\big\rangle \geq 0, \qquad \min_{q\in \operatorname{sign}(Du)} \big\langle (D^2u)q, q\big\rangle \leq 0. \tag{0.4}$$

Here $q \in \operatorname{sign}(p)$ means $q_j = 1$ if $p_j > 0$, $q_j = -1$ if $p_j < 0$ and $q_j \in [-1, 1]$ if $p_j = 0$; the operators involved are now *discontinuous*. There are no known uniqueness results associated with these particular discontinuous equations. Comparison results for absolutely minimizing functions with respect to this - and other - norms is the second topic in this paper. By suitably shielding the discontinuities, in Section 5 we prove uniqueness in a number of cases, including the one above as well as the equation associated with the "l_1 norm" on $\mathbb{R}^n$, namely $|x|_1 = \sum_{l=1}^{n} |x_i|$. Thus uniqueness holds for the l_p norm on $\mathbb{R}^n$ for all $1 \leq p \leq \infty$. Our results for (0.4) make precise a statement about uniqueness made in Belloni and Kawohl [4] without proof.

As regards the presentation, we prefer to give a clean exposition of the basic uniqueness results in unbounded domains in the Euclidean norm before turning to questions associated with other norms. Thus Sections 1 - 4 are written in the Euclidean setting. This will require us to explain how the set-up and arguments in these sections adapt to the more general cases discussed in Section 5 as we launch upon the discussion therein.

We also note here that the results we explicitly present have many generalizations. For example, the results of Sections 3 and 4 extend to the more general norms of Section 5 with some set of details, but we leave this aside. Likewise, our discussion is limited to "absolutely minimizing" functions relative to "Hamiltonians" $H(Du)$ where H is a norm. See, eg, Champion and De Pascale [5], Gariepy, Wang and Yu [9], Yu [15] for basic facts concerning Hamiltonians of more general forms. We have focused our attention on quite new features in the case of norms, namely the approximation theorem, unbounded U and nondifferentiable H. There will be generalizations to other cases, in the settings of the cited papers.

1. Preliminaries and Notation

The symbols U, V, W will always denote open subsets of $\mathbb{R}^n$. $\overline{U}$ is the closure of U and ∂U is its boundary. The notation $V \Subset U$ means that $\overline{V}$ is a compact subset of U.

Regarding balls, we put

$$B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\} \quad \text{and} \quad \overline{B}_r(x) = \overline{B_r(x)}.$$

The Euclidean inner-product is

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n,$$

and, *unless otherwise said*, $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$.

The line segment joining x and y is

$$[x, y] := \{x + t(y-x) : 0 \le t \le 1\}\,; \text{ similarly}$$
$$[x, y) := \{x + t(y-x) : 0 \le t < 1\}\,, \text{ etc.}$$

The notations $A := B$ and $A =: B$ mean, respectively, that A is defined to be B and that B is defined to be A.

If $u : U \to \mathbb{R}$, is smooth enough, then $Du = (u_{x_1}, \dots, u_{x_n})$ is its gradient and D^2u is the Hessian matrix of second derivatives of u. Usually these expressions are understood in the "viscosity sense" when they enter partial differential expressions, and then they do not have pointwise values and are not defined as functions. This will be clear as we proceed.

If $K \subset \mathbb{R}^n$ and $u : K \to \mathbb{R}$, then

$$\operatorname{Lip}(u, K) = \inf\{L \in \mathbb{R} : |u(x) - u(y)| \le L|x-y| \text{ for } x, y \in K\} \tag{1.1}$$

is the least Lipschitz constant for u on K. The convention $\inf \emptyset = \infty$ is employed.

A function $u : U \to \mathbb{R}$ is said to be *absolutely minimizing for* Lip in U provided that

$$V \Subset U, v : \overline{V} \to \mathbb{R}, u|_{\partial V} = v|_{\partial V} \implies \operatorname{Lip}(u, V) \le \operatorname{Lip}(v, V); \tag{1.2}$$

alternatively, since there always exists v satisfying $\operatorname{Lip}(v, \overline{V}) = \operatorname{Lip}(u, \partial V)$, this amounts to $\operatorname{Lip}(u, V) = \operatorname{Lip}(u, \partial V)$ for every $V \Subset U$. The special v's just mentioned are known as the McShane-Whitney extensions ([13], [14], or, for example, [1]).

There are many ways to characterize absolutely minimizing functions. A relevant one for us is: $u : U \to \mathbb{R}$ is absolutely minimizing in U iff

$$\Delta_\infty u := \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0 \text{ in } U. \tag{1.3}$$

The meaning of (1.3) in this paper is the following: suppose $x \in U$ and $\phi \in C^2$ near x. Then

$$\begin{aligned} &\text{(i) } x \text{ is a local maximum of } u - \phi \implies \Delta_\infty \phi(x) \ge 0, \\ &\text{(ii) } x \text{ is a local minimum of } u - \phi \implies \Delta_\infty \phi(x) \le 0. \end{aligned} \tag{1.4}$$

In (1.4), (i) is the definition of $\Delta_\infty u \geq 0$ *in the viscosity sense* and (ii) is the definition of $\Delta_\infty u \leq 0$ *in the viscosity sense.*

Remark 1.1. Throughout this paper $\Delta_\infty u \geq 0, \Delta_\infty u \leq 0, \Delta_\infty u = 0$, are all understood in the viscosity sense without further explanation. If $\Delta_\infty u \geq 0, u$ is said to be ∞-*subharmonic*, while if $\Delta_\infty u \leq 0$, u is said to be ∞-*superharmonic*, and if $\Delta_\infty u = 0$, u is said to be ∞-*harmonic*.

We refer to [6], Section 2, for its proof of the equivalence of "u is absolutely minimizing" and "$\Delta_\infty u = 0$," as we will require proofs from that paper later.

It is shown in [1], Section 4, that $\Delta_\infty u \geq 0$ in U is equivalent

$$x \in B_r(y) \Subset U \implies u(x) - u(y) \leq \frac{\max_{|z-y|=r} u(z) - u(y)}{r}|x-y|. \tag{1.5}$$

This result was first proved by Crandall, Evans and Gariepy, [7].

A key quantity in the proceedings is the "local Lipschitz constant" of $u : U \to \mathbb{R}$ at $x \in U$. It is given by

$$L(u,x) := \lim_{r\downarrow 0} \mathrm{Lip}(u, B_r(x)) = \lim_{r\downarrow 0} \||Du|\|_{L^\infty(B_r(x))}. \tag{1.6}$$

In this generality, the only continuity available for $x \mapsto L(u,x)$ is upper-semicontinuity. The semicontinuity is easy to show, or see [1], Section 1.5.

In the event that u is ∞-subharmonic, then

$$L(u,x) = \lim_{r\downarrow 0} \frac{\max_{|z-y|=r} u(z) - u(x)}{r} \tag{1.7}$$

and the quantity inside the limit on the right is a nondecreasing function of r. Moreover, if $|x-z| = r <$ dist $(x, \partial U)$ (the distance from x to ∂U), $|z-x| = r$ and $u(z) = \max_{\overline{B}_r(x)} u$, then

$$L(u,x) \leq L(u,z). \tag{1.8}$$

We remark that $L(u,x)$ was written $T_u(x)$ in [1].

It was shown in [6], Section 4, that the converse of (1.7) holds. Thus

$$u \text{ is } \infty\text{-subharmonic} \iff L(u,x) \leq \frac{\max_{|z-x|\leq r} u(z) - u(x)}{r} \tag{1.9}$$

for $0 < r <$ dist $(x, \partial U)$. Moreover, if u is ∞-subharmonic, then

$$|Du(x)| = L(u,x) \text{ if } u \text{ is differentiable at } x \in U. \tag{1.10}$$

2. The Approximation Theorem

The main result of this section, indeed, this paper, is the following theorem. See Remark 1.1 regarding conventions used in the statement.

Theorem 2.1. *Let $u \in C(\overline{U})$ be ∞-subharmonic in U. For $\varepsilon > 0$, let $V_\varepsilon = \{x \in U : L(u,x) < \varepsilon\}$. Then there is a family of functions $\{u_\varepsilon\}_{\varepsilon>0}$ with the following properties:*

$$\begin{aligned} &\text{(i) } u_\varepsilon \in C(\overline{U}) \text{ is } \infty\text{-subharmonic in } U,\\ &\text{(ii) } u_\varepsilon = u \text{ on } \overline{U} \setminus V_\varepsilon \text{ and } u_\varepsilon \le u \text{ on } \overline{U},\\ &\text{(iii) } L(u_\varepsilon, x) \ge \varepsilon \text{ for } x \in U,\\ &\text{(iv) } \lim_{\varepsilon \downarrow 0} u_\varepsilon(x) = u(x) \text{ for } x \in U. \end{aligned} \tag{2.1}$$

The proof of Theorem 2.1 will take us a little time, and is presented in stages below. The idea is to generate the u_ε by solving $\varepsilon - |Dw| = 0$ in V_ε subject to $w|_{\partial V_\varepsilon} = u|_{\partial V_\varepsilon}$ by a standard formula and then define $u_\varepsilon = u$ in $\overline{U} \setminus V_\varepsilon$, $u_\varepsilon = w$ in V_ε. Then one has to check that this produces a u_ε with the desired properties. While this description will be useful in figuring out generalizations, we economize here by just using the formulas directly.

The presentation here, stating the full-blown theorem at the outset, is designed to let the reader pass immediately to the following sections to see the applications. The proofs can be read later.

We set about explaining the formula referred to above and verifying some of its properties. In order to write the formula, we consider a general connected open set V and define the "distance" interior to V, as follows. If $x, y \in V$, a path from x to y in V is a continuous mapping

$$\xi : [0,1] \to V \text{ satisfying } \xi(0) = x, \xi(1) = y.$$

We write $\xi \in \text{Path}_V(x,y)$. We will use the elementary description of the length of a path, that is

$$\begin{aligned} &\text{length}(\xi) := \\ &\qquad \sup \left\{ \sum_{i=1}^{m} |\xi(t_i) - \xi(t_{i-1})| : 0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1, m \in Z^+ \right\}. \end{aligned} \tag{2.2}$$

The reason for this "elementary description" is that we wish to render the transition to more general norms in Section 5 as seamless as we can. In fact, this is the notion that generalizes immediately to metric spaces; see, eg, Champion and De Pascale [5].

Fixing $y, x \in V$, the distance from x to y in V is

$$d_V(x,y) := \inf \{\text{length}(\xi) : \xi \in \text{Path}_V(x,y)\}. \tag{2.3}$$

Since V is connected, if $x, y \in V$ there is always a piecewise linear path in V joining x and y, so $d_V(x,y)$ is well-defined and finite. If $y \in \partial V$ and $x \in V$, we define

$$d_V(x,y) := \liminf_{V \ni z \to y} d_V(x,z). \tag{2.4}$$

Note that $d_V(x,y) = \infty$ is possible if $x \in V, y \in \partial V$. It is also clear from the definitions that if $x, \hat{x} \in V$ and $y \in \partial V$, then

$$d_V(x,\hat{x}) = d_V(\hat{x},x) \text{ and } d_V(x,y) \le d_V(\hat{x},y) + d_V(x,\hat{x}). \tag{2.5}$$

It follows that if $y \in \partial V$, then $d_V(x,y)$ is either finite for all $x \in V$ or ∞ for all $x \in V$. We define

$$\mathcal{IN} = \{y \in \partial V : d_V(x,y) = \infty \ \forall x \in V\}, \ \mathcal{AC} = \partial V \setminus \mathcal{IN}, \tag{2.6}$$

and call the points of $\mathcal{IN}$ the *inaccessible* points of ∂V, and the points of $\mathcal{AC}$ the *accessible* points of ∂V. While $\mathcal{IN}$ might be empty or nonempty, $\mathcal{AC}$ is always dense in ∂V. To see this, note that if $y \in \partial V$, $x \in V$ and $|x-y| \le r$, then any point $z \in \partial V$ which is closest to x is in $\mathcal{AC}$ and also satisfies $|z-y| \le |z-x|+|x-y| \le 2r$.

Proposition 2.2. *Let $V \subset \mathbb{R}^n$ be connected, $f \in C(\overline{V})$ satisfy $|f(x)-f(y)| \le |x-y|$ whenever the line segment $[x,y] \subset V$. Then the function w defined in V by*

$$w(x) = \sup_{y \in \partial V} (f(y) - d_V(x,y)) = \sup_{y \in \mathcal{AC}} (f(y) - d_V(x,y)) \tag{2.7}$$

has the following properties:

$$\begin{aligned} &\text{(i) } \textit{if } [y,z] \subset V, \textit{ then } |w(z)-w(y)| \le |z-y|, \\ &\text{(ii) } \textit{if } \overline{B}_r(z) \subset V, \textit{ then } \max_{\{x:|x-z|=r\}} w(x) = w(z) + r. \\ &\text{(iii) } w \le f \textit{ in } V \textit{ and } \lim_{V \ni x \to y} w(x) = f(y) \textit{ for } y \in \partial V, \\ &\text{(iv) } w \textit{ is } \infty\textit{-subharmonic in } V. \end{aligned} \tag{2.8}$$

Remark 2.3. This result is "well-known," but we do not know of a convenient reference with enough detail in just the form we want to use it. It will do no harm to give a clean, self-contained proof here. The property (2.8) (i) guarantees that if $\phi \in C^2$ and $w - \varphi$ has a minimum at $\hat{x}$, then $1 - |D\varphi(\hat{x})| \ge 0$; this is standard. The property (2.8) (ii) guarantees that if $w - \varphi$ has a maximum at $\hat{x}$, then $1 - |D\varphi(\hat{x})| \le 0$. To see this, suppose that $w - \varphi$ has a maximum at $\hat{x}$, so that

$$w(x) - w(\hat{x}) \le \varphi(x) - \varphi(\hat{x}) \text{ for } x \text{ near } \hat{x}.$$

Then, putting $|x - \hat{x}| = r$ and using (2.8) (ii),

$$r = \max_{\overline{B}_r(\hat{x})} w - w(\hat{x}) \le \max_{\overline{B}_r(\hat{x})} \varphi - \varphi(\hat{x}), \tag{2.9}$$

which implies that $|D\varphi(\hat{x})| \ge 1$. That is, together, (i) and (ii) guarantee that w is a viscosity solution of $1 - |Dw| = 0$ in the sense just verified. If $b \in C(\partial V)$, the existence of a viscosity solution of $1 - |Dw| = 0$ in V which satisfies $w = b$ on ∂V requires the existence of an f extending b into $\overline{V}$ with the properties assumed for f. This is a "compatability" condition on the boundary data. See, for example, Lions [12], Chapter 5, Jensen [11], Barron and Jensen [3].

Proof. We establish (2.8) (i). Indeed, for each $y \in \mathcal{A}C$, $x \mapsto f(y) - d_V(x,y)$ has this property because $x \mapsto d_V(x,y)$ satisfies the same condition, as is obvious from the definitions:

$$[x,z] \subset V \implies |d_V(x,y) - d_V(z,y)| \leq |x-z|. \tag{2.10}$$

This establishes (i).

In order to establish (2.8) (ii), it suffices to check that

$$\max_{\{x:|x-z|=r\}} (f(y) - d_V(x,y)) \geq f(y) - d_V(z,y) + r$$

for $y \in \partial V$. Indeed, in "supping" over y to get w, the two sups on the left can be interchanged and the other inequality is guaranteed by (2.8) (i). That is, we only need to check that

$$\hat{y} \in \mathcal{A}C \implies d_V(z,\hat{y}) \geq \min_{\{x:|x-z|=r\}} d_V(x,\hat{y}) + r. \tag{2.11}$$

The final reduction we use is that it suffices to check the conclusion (2.11) for $\hat{y} \in V \setminus \overline{B}_r(z)$ (rather than $\hat{y} \in \mathcal{A}C$). Indeed, suppose the conclusion of (2.11) holds for $\hat{y} \in V \setminus \overline{B}_r(z)$, let $V \ni y^j \to y \in \mathcal{A}C$ and let $x^j \in \partial B_r(z)$ be a point for which

$$d_V(z,y^j) \geq d_V(x^j,y^j) + r.$$

We may assume that $d_V(z,y^j) \to d_V(z,y)$ and $x^j \to x \in \partial B_r(z)$. Since $d_V(x^j,y) \to d_V(x,y)$ uniformly in y by (2.10) (with $z = x^j$), letting $j \to \infty$ above yields

$$d_V(z,y) \geq \limsup_{j\to\infty} d_V(x,y^j) + r \geq d_V(x,y) + r,$$

so (2.11) holds. It remains to check (2.11) for $\hat{y} \in V \setminus \overline{B}_r(z)$. We leave this simple matter to the reader, and provide the hint that any path from $\hat{y}$ to z in V must pierce the sphere $\partial B_r(z)$ at some point x, after which it is rendered "more optimal" by the line segment joining x to z.

We have verified properties (2.8) (i), (ii). The property (2.8) (iv) follows from these. Indeed, from (i) it follows that $L(w,x) \leq 1$ for $x \in V$, and from (ii) it then follows that

$$L(w,x) \leq \frac{\max_{\overline{B}_r(x)} w - w(x)}{r} = 1, \tag{2.12}$$

and this implies that w is ∞-subharmonic ((1.9)). (In fact, $L(w,x) \equiv 1$.)

It remains to show that $w \leq f$ and to study the boundary behavior of w. Let $x,y \in V$, $\xi \in \text{Path}_V(x,y)$. The function f, whose existence is assumed in the proposition, satisfies

$$|f(\xi(t)) - f(\xi(t_{i-1}))| \leq |\xi(t_i) - \xi(t_{i-1})| \tag{2.13}$$

whenever $0 = t_0 < t_1 < \cdots < t_m = 1$ is a sufficiently fine partition of $[0,1]$ so that each interval $[\xi(t_{i-1}), \xi(t_i))]$ lies in V. Thus

$$|f(y) - f(x)| = |f(\xi(1)) - f(\xi(0))| \leq \sum_{i=1}^{m} |\xi(t_i)) - \xi(t_{i-1})| \leq \text{length}(\xi), \tag{2.14}$$

which, in turn, implies that $|f(x)-f(y)| \le d_V(x,y)$ for $x,y \in V$. It then follows from the continuity of f on $\overline{V}$ that

$$f(x) \ge f(y) - d_V(x,y) \text{ for } x \in V, y \in \overline{V}. \tag{2.15}$$

Taking the supremum over $y \in \mathcal{A}C$, we end up with

$$f(x) \ge w(x) \quad \text{for} \quad x \in V. \tag{2.16}$$

Now let $y \in \partial V$, $x \in V$. Let $x^* \in \partial V$ be a nearest point to x. Then $|x^*-x| \le |x-y|$, $|x^*-y| \le |x^*-x| + |x-y| \le 2|x-y|$, which implies

$$\begin{aligned} w(x) \ge f(x^*) - d_V(x,x^*) = f(x^*) - |x-x^*| &\ge f(x^*) - |x-y| \\ &\ge f(y) + (f(x^*) - f(y)) - |x-y| \\ &\ge f(y) + \min_{z\in\partial V, |z-y|\le 2|x-y|} (f(z)-f(y)) - |x-y| \end{aligned} \tag{2.17}$$

It follows from the continuity of f, (2.16) and (2.17) that

$$\lim_{V \ni x \to y} w(x) = f(y)$$

holds for $y \in \partial V$. □

Suppose now that u is ∞-subharmonic in U and $u \in C(\overline{U})$, and let $L(u,x)$ be the local Lipschitz constant of u at x. $L(u,x)$ is upper-semicontinuous in x, so for each $\varepsilon > 0$, $V_\varepsilon = \{x \in U : L(u,x) < \varepsilon\}$ is open in U and hence in $\mathbb{R}^n$. In consequence, using [1], Remark 2.16, $|u(x)-u(y)| \le \varepsilon|x-y|$ whenever $[x,y] \subset V$. Thus, by Proposition 2.2, if V is any component of V_ε, the function

$$\begin{aligned} w_\varepsilon(x) \colon &= \varepsilon \sup_{y\in\partial V} \left(\frac{u(y)}{\varepsilon} - d_V(x,y) \right) \\ &= \sup_{y\in\partial V} (u(y) - \varepsilon d_V(x,y)) \end{aligned}$$

has the properties which can be read off of Proposition 2.2. Having defined w_ε in each component of V_ε and extended it by continuity to $\overline{V_\varepsilon}$, per (2.8) (iii), w_ε is defined and continuous on $\overline{V_\varepsilon}$ and agrees with u on ∂V_ε.

The functions u_ε given by

$$u_\varepsilon(x) := \begin{cases} u(x) \text{ for } x \in \overline{U} \setminus V_\varepsilon \\ w_\varepsilon(x) \text{ for } x \in V_\varepsilon, \end{cases} \tag{2.18}$$

have the properties asserted by Theorem 2.1.

Proof of Theorem 2.1. From the definitions and Proposition 2.2, we have already established the property (2.1) (ii). We next prove (2.1) (iv); namely, $u_\varepsilon(x) \to u(x)$ as $\varepsilon \downarrow 0$. This convergence is clear if $L(u,x) > 0$, for then $u_\varepsilon(x) = u(x)$ for $\varepsilon < L(u,x)$. Hence we need to treat only those x such that $x \in N := \{x \in U : L(u,x) = 0\}$. If $x \in \partial N$, the result holds since u_ε and u are uniformly locally uniformly continuous, which implies that the set of points at which the convergence holds is closed in U. If x is in the interior of N, let $y \in \partial N$ be a nearest point in ∂N to x. Then $u(x) = u(y)$ and $|u_\varepsilon(x) - u_\varepsilon(y)| \leq \varepsilon|x - y|$ since the interval $[x, y) \subset V_\varepsilon$ for $0 < \varepsilon$ (see (2.21) below). Since $u_\varepsilon(y)$ converges to $u(y)$, we are done with (iv).

We turn to the proof of (2.1) (i); namely, u_ε is ∞-subharmonic. The somewhat subtle point of the proof in our view is this:

$$x \in (\partial V_\varepsilon) \cap U \implies L(u_\varepsilon, x) \leq L(u, x). \tag{2.19}$$

Since V_ε is open, if $x \in (\partial V_\varepsilon) \cap U$, then $x \notin V_\varepsilon$, and we have, by the definition of V_ε, $L(u, x) \geq \varepsilon$, which renders the ε's appearing in estimates appearing below harmless as regards establishing (2.19).

To establish (2.19), it suffices to show that if $[y, z] \subset B_r(x) \Subset U$, then

$$|u_\varepsilon(z) - u_\varepsilon(y)| \leq \mathrm{Lip}(u, B_r(x))|y - z|, \tag{2.20}$$

for then $\mathrm{Lip}(u_\varepsilon, B_r(x)) \leq \mathrm{Lip}(u, B_r(x))$.

First note that for any open interval $(y, z) \subset V_\varepsilon$, then Proposition 2.2 (i), continuity, and the definitions provide the estimate

$$|u_\varepsilon(z) - u_\varepsilon(y)| \leq \varepsilon|y - z|. \tag{2.21}$$

In particular, if $[y, z] \subset V_\varepsilon$, then (2.20) holds. If $[y, z] \cap (U \setminus V_\varepsilon)$ is not empty, we proceed as follows. The set $U \setminus V_\varepsilon$ is closed in U, so $[y, z] \cap (U \setminus V_\varepsilon)$ is closed and there exists a least $t_0 \in [0, 1]$ and a greatest $t_1 \in [0, 1]$ such that

$$x^0 := y + t_0(z - y) \in U \setminus V_\varepsilon, \ \ x^1 := y + t_1(z - y) \in U \setminus V_\varepsilon. \tag{2.22}$$

Then

$$\begin{aligned} |u_\varepsilon(z) - u_\varepsilon(y)| &= |u_\varepsilon(z) - u_\varepsilon(x^1) + u_\varepsilon(x^1) - u_\varepsilon(x^0) + u_\varepsilon(x^0) - u_\varepsilon(y)| \\ &= |u_\varepsilon(z) - u_\varepsilon(x^1) + u(x^1) - u(x^0) + u_\varepsilon(x^0) - u_\varepsilon(y)| \\ &\leq |u_\varepsilon(z) - u_\varepsilon(x^1)| + |u(x^1) - u(x^0)| + |u_\varepsilon(x^0) - u_\varepsilon(y)| \\ &\leq |u_\varepsilon(z) - u_\varepsilon(x^1)| + \mathrm{Lip}(u, B_r(x))|x^1 - x^0| + |u_\varepsilon(x^0) - u_\varepsilon(y)| \end{aligned} \tag{2.23}$$

Now each interval $[y, x^0)$ and $(x^1, z]$ is either empty (as is the case, for example, for $[y, x^0)$ if $y = x^0$) or lies entirely in V_ε. Using (2.21) twice, with $z = x^0$ and with $y = x^1$,

$$|u_\varepsilon(z) - u_\varepsilon(x^1)| \leq \varepsilon|z - x^1|, \ \ |u_\varepsilon(x^0) - u_\varepsilon(y)| \leq \varepsilon|x^0 - y|.$$

Combining this with (2.23), (2.20) follows from

$$|z - y| = |z - x^1| + |x^1 - x^0| + |x^0 - y|.$$

We continue. If we can show that

$$L(u_\varepsilon, x) \le \max_{\{y:|y-z|\le r\}} \left(\frac{u_\varepsilon(y) - u_\varepsilon(x)}{r}\right) \tag{2.24}$$

for $r < \mathrm{dist}\,(x, \partial U)$, then u_ε is ∞-subharmonic. If $x \notin V_\varepsilon$, then (2.24) holds. Indeed, then, using (2.19) if necessary (i.e, if $x \in \partial V_\varepsilon$), we have

$$L(u_\varepsilon, x) \le L(u, x) \le \frac{u(y_r) - u(x)}{r} = \frac{u_\varepsilon(y_r) - u_\varepsilon(x)}{r} \tag{2.25}$$

for some y_r, $|y_r - x| = r$, which satisfies $u(y_r) = \max_{\overline{B}_r(x)} u$ and therefore also $L(u, y_r) \ge L(u, x) \ge \varepsilon$ (see (1.8)). Since $y_r \notin V_\varepsilon$, we have $u(y_r) = u_\varepsilon(y_r)$. Hence (2.24) holds if $x \notin V_\varepsilon$.

To handle the case $x \in V_\varepsilon$, one first observes that (2.8) (i), (ii) imply that if $B_r(z) \subset V_\varepsilon$ then

$$\max_{y \in \overline{B}_r(z)} u_\varepsilon(y) = u_\varepsilon(z) + \varepsilon r, \tag{2.26}$$

and $L(u_\varepsilon, z) = \varepsilon$. Recalling that $L(u, x) \ge \varepsilon$ for $x \in U \setminus V_\varepsilon$, between (2.25) and (2.26), we learn that for every $z \in U$ there is an $r_z > 0$ such that

$$\max_{y \in \overline{B}_r(z)} u_\varepsilon(y) \ge u_\varepsilon(z) + \varepsilon r \text{ for } 0 \le r \le r_z.$$

This implies, with a little continuation argument, that

$$\varepsilon \le \max_{\{y:|y-x|\le r\}} \left(\frac{u_\varepsilon(y) - u_\varepsilon(x)}{r}\right) \text{ for } r < \mathrm{dist}\,(x, \partial U),$$

and we are done; u_ε is ∞-subharmonic. Moreover, along the way, we have shown that the last estimate holds for every $x \in U$, whence (2.1) (iii). □

3. Uniqueness When ∂U is Bounded

In this section we treat the case in which ∂U is bounded. This splits into two subcases: either U is bounded or U is unbounded. As explained in the introduction, comparison for the case in which U is bounded was first settled by Jensen [11] and then a different proof was given by Barles and Busca [2]. We offer a third proof here, relying on Theorem 2.1. Then we show how to reduce the case in which U is unbounded, the "exterior problem," to the previous result. Again, this uses Theorem 2.1. We turn to the proof of

Theorem 3.1. *Let U be bounded. Let $u, v \in C(\overline{U})$, and $\Delta_\infty u \ge 0$, $\Delta_\infty v \le 0$ in U. Then*

$$u(x) - v(x) \le \max_{\partial U}(u - v) \text{ for } x \in U. \tag{3.1}$$

Proof. It suffices to establish (3.1) with u replaced by the u_ε of Theorem 2.1. Thus we may assume that $L(u, x) \ge \varepsilon$ for $x \in U$, where $\varepsilon > 0$. Let $\lambda > 0$ and $\sup_U u < \frac{1}{2\lambda}$ and then define w by

$$u(x) = w(x) - \frac{\lambda}{2} w(x)^2 \text{ and } w(x) \le \frac{1}{\lambda}. \tag{3.2}$$

That is, $w(x) = (1 - \sqrt{1 - 2\lambda u})/\lambda$. Since u is bounded and $w \to u$ as $\lambda \downarrow 0$, uniformly on sets on which u is bounded, it suffices to show that (3.1) holds with w in place of u. This is a standard change of variables, see [11].

We formally calculate the differential inequality w satisfies as follows:

$$Du = (1 - \lambda w)Dw, \; D^2u = -\lambda Dw \otimes Dw + (1 - \lambda w)D^2 w,$$

which implies that

$$\begin{aligned} 0 \le \Delta_\infty u :&= \left\langle D^2uDu, Du\right\rangle \\ &= \left\langle(-\lambda Dw \otimes Dw + (1 - \lambda w)D^2 w)(1 - \lambda w)Dw, (1 - \lambda w)Dw\right\rangle \\ &= -\lambda(1 - \lambda w)^2|Dw|^4 + (1 - \lambda w)^3 \left\langle D^2wDw, Dw\right\rangle \\ &= -\lambda(1 - \lambda w)^{-2}|Du|^4 + (1 - \lambda w)^3 \left\langle D^2wDw, Dw\right\rangle \end{aligned} \tag{3.3}$$

Thus, recalling $L(u, x) \ge \varepsilon$,

$$\frac{\lambda}{(1 - \lambda w)^5}\varepsilon^4 \le \Delta_\infty w. \tag{3.4}$$

All of these calculations are correct in the viscosity sense, and this is standard, up to the appearance of $|Du|$ and then its estimation by ε via $L(u, x) \ge \varepsilon$. Since w is ∞-subharmonic by the "correct" part of the calculations, the additional fact we need is this: if w is ∞-subharmonic and $w - \phi$ has a local maximum at $\hat{x} \in U$, then

$$L(w, \hat{x}) \le |D\varphi(\hat{x})|. \tag{3.5}$$

However, using the inequality of (2.9) divided by r, we have

$$\frac{\max_{\overline{B}_r(\hat{x})} w - w(\hat{x})}{r} \le |D\varphi(\hat{x})| + \mathrm{o}(1)$$

as $r \downarrow 0$. Finally, if G is C^1, then it is easy to see that if $u = G(w)$ one has $L(u, x) = |G'(w(x))|L(w, x)$. Altogether, this justifies (3.4). Now, for some $\gamma > 0$ and all small enough λ, (3.4) implies

$$0 < \gamma \le \Delta_\infty w \text{ in } U \tag{3.6}$$

where γ depends only on λ.

It is now completely standard that (3.1) holds with w in place of u. We review this. Consider

$$\max_{\overline{U}\times\overline{U}} \Psi_\alpha(x, y) \text{ where } \Psi_\alpha(x, y) := w(x) - v(y) - \frac{\alpha}{2}|x - y|^2. \tag{3.7}$$

The maximum is assumed at some point (x_α, y_α):

$$M_\alpha := \max_{\overline{U}\times\overline{U}} \Psi_\alpha = w(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2. \tag{3.8}$$

In addition, we define

$$M_0 := \sup_{\overline{U}}(w(x) - v(x)). \tag{3.9}$$

By Lemma 3.1 of [8]

$$\text{(3.10)}\qquad \text{(i)} \lim_{\alpha\to\infty} M_\alpha = M_0,\ \text{(ii)} \lim_{\alpha\to\infty} \alpha|x_\alpha - y_\alpha|^2 = 0,\ \text{(iii)} \lim_{\alpha\to\infty}(w(x_\alpha) - v(y_\alpha)) = M_0.$$

Assume that $\max_{\overline{U}}(w-v) > \max_{\partial U}(w-v)$. It then follows from (3.10) that $x_\alpha, y_\alpha \in U$ for large α.

Now apply Theorem 3.2 of [8] and the discussion following the theorem to assert the existence of $n\times n$ real symmetric matrices X, Y, such that

$$\text{(3.11)}\qquad -3\alpha \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \le \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \le 3\alpha \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}$$

and, recalling (3.6),

$$\gamma < \alpha^2 \langle X(x_\alpha - y_\alpha), x_\alpha - y_\alpha\rangle,\ \alpha^2 \langle Y(x_\alpha - y_\alpha), x_\alpha - y_\alpha\rangle \le 0.$$

Since $X \le Y$ by (3.11), this is impossible. □

We use Theorem 2.1 to prove:

Theorem 3.2. *Let U be unbounded and ∂U be bounded. Let $u, v \in C(\overline{U})$, and $\Delta_\infty u \ge 0$, $\Delta_\infty v \le 0$ in U. Assume also that*

$$\text{(3.12)}\qquad \limsup_{|x|\to\infty} \frac{u(x)}{|x|} \le 0 \text{ and } \liminf_{|x|\to\infty} \frac{v(x)}{|x|} \ge 0,$$

Then

$$\text{(3.13)}\qquad u(x) - v(x) \le \max_{\partial U}(u-v) \quad \text{for } x \in U.$$

Proof. Let $\varepsilon > 0$ and u_ε be as in Theorem 2.1. It suffices to prove that $u_\varepsilon \le v$. We show below that

$$\text{(3.14)}\qquad \limsup_{|x|\to\infty} \frac{u_\varepsilon(x)}{|x|} \le -\varepsilon$$

Using the assumption (3.12) as regards v and the above, it follows that

$$u_\varepsilon(x) - v(x) < -\frac{\varepsilon}{2}|x| \text{ for large } |x|.$$

Hence, applying Theorem 3.1, for large R we have

$$u_\varepsilon(x) - v(x) \le \max_{\partial(U\cap B_R(0))}(u-v) = \max_{\partial U}(u-v) \quad \text{for } x \in U\cap B_R(0)\,.$$

Now let $R\to\infty$.

We verify (3.14). The function

$$r \mapsto g(r) := \max_{|x|=r} u_\varepsilon$$

is convex on the set of r such that $\partial U \subset B_r$. See the proof of Lemma 2.6 in [1] (which works more generally than cited in that lemma). By (3.12) for u and $u_\varepsilon \le u$, it must be that

$$k := \lim_{r\to\infty} \frac{g(r)}{r} \le 0.$$

However, by (1.7), the monotonicity of the right-hand side of (1.7) in r, and simple considerations,

$$\begin{aligned} \varepsilon \le L(u_\varepsilon, x) \le \frac{\max_{|y-x|=r} u_\varepsilon(y) - u_\varepsilon(x)}{r} &\le \frac{\max_{|x|-r\le|y|\le r+|x|} u_\varepsilon(y) - u_\varepsilon(x)}{r} \\ &\le \max_{|x|-r\le s\le r+|x|} \frac{g(s)}{r} - \frac{u(x)}{r} \end{aligned} \tag{3.15}$$

provided that $\overline{B}_r(x) \subset U$ and $\partial U \subset B_{r-|x|}$. Choose $r = \alpha|x|$, $0 < \alpha < 1$, and let $|x| \to \infty$ to find

$$\varepsilon \le k\frac{1-\alpha}{\alpha} - \limsup_{|x|\to\infty} \frac{u(x)}{\alpha|x|} \le k\frac{1-\alpha}{\alpha} - \frac{k}{\alpha} = -k$$

The result (3.14) follows then from the definitions of g, k. □

4. THE CASE OF UNBOUNDED ∂U

4.1. **General ∂U.** We do not know whether or not sublinear growth itself is enough to guarantee the comparison theorem when U and ∂U are both unbounded. However, the result below does hold.

Theorem 4.1. *Let U be an open subset of $\mathbb{R}^n$, $u, v : \overline{U} \to \mathbb{R}$ be uniformly continuous, bounded, and satisfy $\Delta_\infty u \ge 0$, $\Delta_\infty v \le 0$ in U. Then*

$$\sup_U (u - v) \le \sup_{\partial U} (u - v). \tag{4.1}$$

The proof runs similarly to that of Theorem 3.1, but we will have to take care of ∞ in a more complicated way than the elegant method used to prove Theorem 3.2. However, the general idea of penalizing ∞ as done below is standard. The devil is in the details.

Proof. By Theorem 2.1 we may assume that

$$0 < \varepsilon \le L(u, x) \text{ for } x \in U, \tag{4.2}$$

although we lose the assumption that u is bounded. However, this "new u" will still be bounded above, bounded on ∂U, and uniformly continuous (as follows from estimates like (2.23), without the last line of (2.23)). We assume that (4.1) does not hold, that is,

$$\sup_{\partial U} (u - v) < \sup_U (u - v). \tag{4.3}$$

Define w by (3.2) as before. Since u is bounded on ∂U and $w \to u$ as $\lambda \downarrow 0$, uniformly on sets on which u is bounded, (4.3) implies that

$$\sup_{\partial U} (w - v) < \sup_U (w - v) \tag{4.4}$$

if λ is small. It is still true that (3.4) holds, that is

$$\frac{\lambda}{(1-\lambda w)^5}\varepsilon^4 \le \Delta_\infty w. \tag{4.5}$$

We proceed to show that (4.5) is incompatible with (4.4). To this end, we penalize ∞ via a linearly growing function $P \ge 0$ which has a bounded gradient and Hessian, for example,

$$P(x) = \sqrt{|x|^2+1}, \tag{4.6}$$

and consider

$$\max_{\overline{U}\times\overline{U}} \Psi_\alpha(x,y) \text{ where } \Psi_\alpha(x,y) := w(x) - v(y) - \frac{\alpha}{2}|x-y|^2 - \frac{1}{\alpha^2}P(y). \tag{4.7}$$

Since $w - v$ is bounded above, the maximum is assumed at some point (x_α, y_α):

$$M_\alpha := \max_{\overline{U}\times\overline{U}} \Psi_\alpha = w(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 - \frac{1}{\alpha^2}P(y_\alpha). \tag{4.8}$$

In addition, we define

$$M_0 := \sup_{\overline{U}}(w(x) - v(x)),\ \tilde{M}_\alpha := \sup_{\overline{U}\times\overline{U}} \left(w(x) - v(y) - \frac{\alpha}{2}|x-y|^2\right). \tag{4.9}$$

We claim that:

$$\begin{aligned} &\text{(i)}\ \lim_{\alpha\to\infty} M_\alpha = M_0,\ \text{(ii)}\ \lim_{\alpha\to\infty}\left(\frac{\alpha}{2}|x_\alpha - y_\alpha|^2 + \frac{1}{\alpha^2}P(y_\alpha)\right) = 0,\\ &\text{(iii)}\ \lim_{\alpha\to\infty}(w(x_\alpha) - v(y_\alpha)) = M_0. \end{aligned} \tag{4.10}$$

To establish (i), note that, via (4.8), (4.9),

$$\tilde{M}_\alpha \ge M_\alpha \ge \sup_{\overline{U}}(w(x) - v(x) - \frac{1}{\alpha^2}P(x)).$$

If we choose x so that $w(x) - v(x) \ge M_0 - \delta$ and freeze it, this yields

$$\tilde{M}_\alpha \ge M_\alpha \ge M_0 - \delta - \frac{1}{\alpha^2}P(x);$$

now let $\alpha \to \infty$ to find

$$M_0 = \lim_{\alpha\to\infty}\tilde{M}_\alpha \ge \limsup_{\alpha\to\infty} M_\alpha \ge \liminf_{\alpha\to\infty} M_\alpha \ge M_0 - \delta.$$

The first identity above is from [8], Lemma 3.1. Letting $\delta \downarrow 0$, we have proved 4.10 (i). Since(4.8) also implies

$$M_\alpha \le \tilde{M}_\alpha - \frac{1}{\alpha^2}P(y_\alpha),$$

it must be that $P(y_\alpha)/\alpha^2 \to 0$ as $\alpha \to \infty$. But then (4.8) and (4.10) (i) imply that

$$\lim_{\alpha\to\infty}\left(w(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2\right) = M_0.$$

Now (4.10) (ii) and (iii) follow from [8] Lemma 3.1.

A consequence of all this is the following:

$$\liminf_{\alpha\to\infty} \operatorname{dist}(x_\alpha, \partial U) > 0,\ \liminf_{\alpha\to\infty} \operatorname{dist}(y_\alpha, \partial U) > 0. \tag{4.11}$$

We argue informally. Since w, v are uniformly continuous, if x_α is sufficiently close to ∂U and y_α is sufficiently close to x_α (as it will be, by (4.10) (ii)), we can guarantee that $w(x_\alpha) - v(y_\alpha)$ is as close as we please to $w(x_\alpha) - v(x_\alpha)$, which is as close as we please to the value of $w - v$ at the point of ∂U nearest x_α. But then $w(x_\alpha) - v(x_\alpha)$ cannot be close to M_0 (by (4.4)), contradicting (4.10) (iii).

In view of (4.11), for large α we may apply Theorem 3.2 of [8] (with the ε of that theorem set equal to $1/\alpha$) to the data provided by (4.8) to assert, using (4.5), that there are matrices X, Y such that

$$\begin{aligned} &\frac{\lambda\varepsilon^4}{(1-\lambda w(x_\alpha))^5} \le \alpha^2 \left\langle X\left(x_\alpha - y_\alpha\right), x_\alpha - y_\alpha \right\rangle, \\ &\left\langle Y\left(\alpha(x_\alpha - y_\alpha) - \frac{1}{\alpha^2}DP(y_\alpha)\right), \alpha(x_\alpha - y_\alpha) - \frac{1}{\alpha^2}DP(y_\alpha)\right\rangle \le 0 \end{aligned} \tag{4.12}$$

and

$$-\left(\alpha I + \|A\|\right) I \le \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \le A + \frac{1}{\alpha}A^2 \tag{4.13}$$

where

$$A = \alpha \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + \frac{1}{\alpha^2}\begin{bmatrix} 0 & 0 \\ 0 & D^2P(y_\alpha) \end{bmatrix}. \tag{4.14}$$

We won't need the lower bound of (4.13). Note that

$$A^2 = 2\alpha^2 \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + \frac{1}{\alpha}Z(\alpha)$$

where $Z(\alpha)$ remains bounded as $\alpha \to \infty$. Thus

$$A + \frac{1}{\alpha}A^2 = 3\alpha \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + \frac{1}{\alpha^2}Z(\alpha) \tag{4.15}$$

where Z is another bounded function of α. We set

$$p_\alpha = \alpha(x_\alpha - y_\alpha). \tag{4.16}$$

Using (4.13), (4.15), we find

$$\begin{aligned} &\left\langle \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \begin{bmatrix} p_\alpha \\ p_\alpha - \frac{1}{\alpha^2}DP(y_\alpha) \end{bmatrix}, \begin{bmatrix} p_\alpha \\ p_\alpha - \frac{1}{\alpha^2}DP(y_\alpha) \end{bmatrix} \right\rangle \\ &\quad \le \left\langle \left(3\alpha \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + \frac{1}{\alpha^2}Z(\alpha)\right) \begin{bmatrix} p_\alpha \\ p_\alpha - \frac{1}{\alpha^2}DP(y_\alpha) \end{bmatrix}, \begin{bmatrix} p_\alpha \\ p_\alpha - \frac{1}{\alpha^2}DP(y_\alpha) \end{bmatrix} \right\rangle \\ &= 3\frac{1}{\alpha^3}|DP(y_\alpha)|^2 + \frac{1}{\alpha^2}\left\langle Z(\alpha) \begin{bmatrix} p_\alpha \\ p_\alpha - \frac{1}{\alpha^2}DP(y_\alpha) \end{bmatrix}, \begin{bmatrix} p_\alpha \\ p_\alpha - \frac{1}{\alpha^2}DP(y_\alpha) \end{bmatrix} \right\rangle. \end{aligned}$$

Since $|p_\alpha|^2/\alpha^2 = |x_\alpha - y_\alpha|^2 \to 0$ as $\alpha \to \infty$ by (4.10), this inequality implies that

$$\langle Xp_\alpha, p_\alpha\rangle - \left\langle Y(p_\alpha - \frac{1}{\alpha^2}DP(y_\alpha)), p_\alpha - \frac{1}{\alpha^2}DP(y_\alpha)\right\rangle \le C\left(\frac{1}{\alpha^2} + |x_\alpha - y_\alpha|\right) \to 0$$

as $\alpha \to \infty$. However, this contradicts (4.12), provided that $w(x_\alpha)$ remains bounded below as $\alpha \to \infty$. In view of (4.10) (iii) and the assumption that v is bounded, this is indeed the case. □

4.2. **The Half - Space.** We first consider the case of the upper-half space:

$$U = \mathcal{H} = \{x \in \mathbb{R}^n : x_n > 0\}. \tag{4.17}$$

In this case, we set

$$x' = (x_1, \dots, x_{n-1}) \text{ so that } x = (x', x_n).$$

For $T > 0$, let $\mathcal{H}_T$ be the strip

$$\mathcal{H}_T = \left\{(x', x_n) \in \overline{\mathcal{H}} : 0 \le x_n \le T\right\}. \tag{4.18}$$

Theorem 4.2. *Let $u, -v$ be ∞-subharmonic in $\mathcal{H}$, uniformly continuous and bounded in each strip $\mathcal{H}_T, 0 < T$, and satisfy*

$$\limsup_{x_n \to \infty} \frac{u(x', x_n)}{x_n} \le 0, \ \liminf_{x_n \to \infty} \frac{v(x', x_n)}{x_n} \ge 0 \tag{4.19}$$

uniformly in $x' \in \mathbb{R}^{n-1}$. Then

$$u(x) - v(x) \le \sup_{\partial \mathcal{H}} (u - v) \text{ for } x \in \mathcal{H}. \tag{4.20}$$

Proof. We use Theorem 4.1 and mimic parts of the proof of Theorem 3.1. Replacing u by u_ε, per Theorem 2.1, we preserve all of the assumptions of the theorem, and, in addition, we may assume that $L(u, x) \ge \varepsilon$. This new u is still uniformly continuous in each strip, and therefore bounded in each strip, as it is bounded on $\partial\mathcal{H}$.

It now follows from Theorem 4.1 that

$$u(x', x_n) \le \sup_{y' \in \mathbb{R}^{n-1}} u(y', R) \left(\frac{x_n - r}{R - r}\right) + \sup_{y' \in \mathbb{R}^{n-1}} u(y', r) \left(\frac{R - x_n}{R - r}\right)$$

for $0 \le r \le x_n \le R$. Indeed, the right-hand side is ∞-harmonic, and the inequality holds on $\partial(\mathcal{H}_R \setminus \mathcal{H}_r)$ and hence in $\mathcal{H}_R \setminus \mathcal{H}_r$. That is,

$$g(x_n) := \sup_{x' \in \mathbb{R}^{n-1}} u(x', x_n) \text{ is convex on } 0 \le x_n. \tag{4.21}$$

By (4.19),

$$\lim_{x_n \to \infty} \frac{g(x_n)}{x_n} =: k \le 0. \tag{4.22}$$

A simple adaptation of (3.15) to use strips here yields $k \le -\varepsilon$. Thus $u - v \le \sup_{\partial\mathcal{H}}(u - v)$ on $\partial\mathcal{H}_T$ if T is large enough, and we conclude as in the proof of Theorem 3.2. □

5. Uniqueness for Other Norms

5.1. **Preliminaries.** In this section $|\cdot|$ denotes, for a while, *an arbitrary* norm on $\mathbb{R}^n$. While the preceding text assumed that $|\cdot|$ was the Euclidean norm on $\mathbb{R}^n$, much of it applies equally well to the general case.

In particular, in Sections 1 and 2 the changes described below are all that are necessary to allow $|\cdot|$ to be a general norm.

The definitions of balls in Section 1 remain in effect, but now use $|\cdot|$, and we also put

$$B = \{q \in \mathbb{R}^n : |q| \le 1\} = \overline{B}_1(0) \quad \text{and} \quad S = \{q \in \mathbb{R}^n : |q| = 1\} \tag{5.1}$$

for the unit ball and sphere with respect to the norm $|\cdot|$. Sometimes we will want to refer to the Euclidean norm as well, it will be denoted by

$$|x|_2 = |(x_1, \dots, x_n)|_2 = \sqrt{x_1^2 + \cdots + x_n^2}. \tag{5.2}$$

We need the dual norm and the associated duality mapping in order to describe the necessary changes. The norm dual to $|\cdot|$ is defined, as usual, by

$$|x|^* = \max_{\{y:|y|=1\}} \langle x, y \rangle \,. \tag{5.3}$$

Clearly, then

$$|\langle x, y \rangle| \le |x|^* |y|. \tag{5.4}$$

Next, given $x \in \mathbb{R}^n$, we define

$$J(x) = \{y \in \mathbb{R}^n : |y| \le 1 \text{ and } \langle x, y \rangle = |x|^*\} \,. \tag{5.5}$$

We call J the "duality map" (from $(\mathbb{R}^n, |\cdot|^*)$ to $(\mathbb{R}^n, |\cdot|)$). It is easy to see that $J(x)$ is always a nonempty, closed and convex set. For example, $J(0) = \overline{B_1(0)}$ no matter what the norm is, while if $|\cdot|$ is the Euclidean norm and $x \ne 0$, then $J(x) = \{x/|x|\}$. In general, by (5.4), if $x \ne 0$ then $|y| = 1$ if $y \in J(x)$.

If, as in the Euclidean case, $J(x)$ is "single-valued" for $x \ne 0$, that is, it is a singleton, we do not distinguish between the set $J(x)$ and its element.

The proof of [6], Section 2, that if u is absolutely minimizing, then it necessarily satisfies (1.3) in the sense (1.4) easily modifies (see also [1], Section 5) to show that, in the case of a general norm,

$$\begin{aligned} &\text{(i) } x \text{ is a local maximum of } u - \phi \implies \max_{q \in J(D\phi(x))} \left\langle D^2\phi(x)q, q \right\rangle \ge 0, \\ &\text{(ii) } x \text{ is a local minimum of } u - \phi \implies \min_{q \in J(D\phi(x))} \left\langle D^2\phi(x)q, q \right\rangle \le 0. \end{aligned} \tag{5.6}$$

Moreover, an example is given in [1] to show that one cannot drop the "max" or the "min" above.

The presentation in Section 2 was designed so that it does not depend on Euclidean properties of the norm, and the u_ε's of that section are available in the general case. A primary difference is that references to "∞-subharmonic, ∞-superharmonic, ∞-harmonic" should be replaced by, respectively, "enjoys comparison with cones from above, enjoys comparison with cones from below, and enjoys comparison with cones." These concepts are primary in [1]. One can also use other of the various equivalences of [1], Section 4.

We are able, via our u_ε's, to avoid "zeros of gradients," so the technical difficulty we are focusing on here is the discontinuity of the maps

$$\text{(5.7)}\qquad \begin{aligned} &\text{(i) } (X,p) \mapsto \max_{q\in J(p)} \langle Xq, q\rangle =: F^+(X,p) \text{ for } p \in \mathbb{R}^n \setminus \{0\}\,,\\ &\text{(ii) } (X,p) \mapsto \min_{q\in J(p)} \langle Xq, q\rangle =: F^-(X,p), \text{ for } p \in \mathbb{R}^n \setminus \{0\}\,, \end{aligned}$$

where X is a symmetric $n \times n$ matrix. These mappings are not continuous at (X,p) only if $J(p)$ is not a singleton. The requirement (5.6) is the definition of $F^+(D^2u, Du) \ge 0$ (in the viscosity sense), and (5.6) (ii) is the definition of $F^-(D^2u, Du) \le 0$. We do not have available results of the form "$F^+(D^2u, Du) \ge 0$ implies that u enjoys comparison with cones from above," or "$F^+(D^2u, Du) \ge 0$ and $F^-(D^2u, Du) \le 0$ implies that u is absolutely minimizing for $|\cdot|$" in general. The results herein will establish these assertions in some new cases. Please note that

$$\text{(5.8)}\qquad F^+ \text{ is upper semicontinuous and } F^- \text{ is lower semicontinuous.}$$

Consider $u, v \in C(\overline{U})$ which satisfy $F^+(D^2u, Du) \ge 0$ and $F^-(D^2v, Dv) \le 0$ in U in the sense (5.6). Assuming that U is bounded, we attempt to prove that

$$\text{(5.9)}\qquad u - v \le \max_{\partial U}(u - v) \text{ in } U.$$

It is enough to prove this with u replaced by u_ε, which satisfies $L(u_\varepsilon, x) \ge \varepsilon$ in U. (See the proof of Theorem 5.2 below.) We may still use the approximation w of u given by (3.2), now under the assumption $L(u_\varepsilon, x) \ge \varepsilon$. The calculation (3.3) is replaced by

$$\text{(5.10)}\qquad \begin{aligned} 0 &\le \max_{q\in J(Dw)} \left\langle (D^2u)q, q\right\rangle\\ &= \max_{q\in J(Dw)} \left\langle (-\lambda Dw \otimes Dw + (1-\lambda w)D^2w)(1-\lambda w)q, (1-\lambda w)q\right\rangle\\ &= -\lambda(1-\lambda w)^2(|Dw|^*)^2 + (1-\lambda w)^3 \max_{q\in J(Dw)} \left\langle (D^2w)q, q\right\rangle\\ &= -\lambda(|Du|^*)^2 + (1-\lambda w)^3 \max_{q\in J(Dw)} \left\langle (D^2w)q, q\right\rangle, \end{aligned}$$

where we used $|Dw|^* = \langle Dw, q\rangle$ for $q \in J(Dw)$. Again, $L(u,x) \ge \varepsilon$ allows us to replace $|Du|^*$ by ε above, so, in all,

$$\text{(5.11)}\qquad 0 < \gamma \le \max_{q\in J(Dw)} \left\langle (D^2w)q, q\right\rangle,$$

holds in the viscosity sense. Thus, to prove 5.9, we can assume that (5.11) holds for u, in the obvious viscosity sense.

We adapt the strategy of Section 3. Suppose that $P : \mathbb{R}^n \to \mathbb{R}$ is a C^2 "penalty function" for $|\cdot|$. This will mean that P has the following properties:

$$\text{(5.12)}\qquad \begin{aligned} &\text{(i)} P(x) > P(0) \text{ for } x \ne 0, \text{ and } P(0) = 0,\\ &\text{(ii) if } DP(x) \ne 0 \text{ and } q, \hat{q} \in J(DP(x)), \text{ then } \left\langle D^2P(x)(q-\hat{q}), q-\hat{q}\right\rangle \le 0. \end{aligned}$$

Remark 5.1. If J is single-valued on $\mathbb{R}^n \setminus \{0\}$, which is the case exactly when $|\cdot|$ is strictly convex, then $P(x) = |x|_2^2$ satisfies (5.12).

A slight generalization of Lemma 3.1 of [8] then tells us that if

$$\begin{aligned} M_\alpha :&= \max_{\overline{U}\times\overline{U}} \left(u(x) - v(y) - \alpha P(x-y) \right), \\ &= u(x_\alpha) - v(y_\alpha) - \alpha P(x_\alpha - y_\alpha), \end{aligned} \tag{5.13}$$

then

$$M_\alpha \downarrow M_0 := \max_{\overline{U}\times\overline{U}} (u(x) - v(x)), \ \lim_{\alpha\to\infty} \alpha P(x_\alpha - y_\alpha) = 0. \tag{5.14}$$

It follows that

$$\lim_{\alpha\to\infty} |x_\alpha - y_\alpha| = 0. \tag{5.15}$$

Assuming that (5.9) fails, it then follows from (5.12) (i), (5.14) and (5.15) that if α is sufficiently large, then $(x_\alpha, y_\alpha) \in U \times U$. We fix α so that this is true and denote (x_α, y_α) by $(\hat{x}, \hat{y})$.

Since $(x, y) \mapsto u(x) - v(y) - \alpha P(x - y)$ has a maximum at $(x_\alpha, y_\alpha) = (\hat{x}, \hat{y})$, Theorem 3.2, [8] (with ε replaced by ε/α^2) implies the existence of $p \in \mathbb{R}^n$ and symmetric matrices X, Y such that

$$\begin{aligned} &p = \alpha DP(\hat{x} - \hat{y}) \text{ and } \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \\ &\alpha \begin{bmatrix} D^2P(\hat{x}-\hat{y}) & -D^2P(\hat{x}-\hat{y}) \\ -D^2P(\hat{x}-\hat{y}) & D^2P(\hat{x}-\hat{y}) \end{bmatrix} + 2\varepsilon \begin{bmatrix} D^2P(\hat{x}-\hat{y})^2 & -D^2P(\hat{x}-\hat{y})^2 \\ -D^2P(\hat{x}-\hat{y})^2 & D^2P(\hat{x}-\hat{y})^2 \end{bmatrix}, \end{aligned} \tag{5.16}$$

and for some $q, \hat{q} \in J(p)$,

$$\gamma \leq \langle Xq, q \rangle, \ \langle Y\hat{q}, \hat{q} \rangle \leq 0. \tag{5.17}$$

The fact (5.8) is what is needed to verify (5.17). Furthermore, $|p|^* \geq \varepsilon$.

Using (5.16) above, we find, computing the corresponding values of the associated quadratic forms at $\begin{bmatrix} q \\ \hat{q} \end{bmatrix}$, that

$$\begin{aligned} \langle q, Xq \rangle \leq \langle \hat{q}, Y\hat{q} \rangle + \alpha \left\langle D^2P(\hat{x}-\hat{y})(q-\hat{q}), q-\hat{q} \right\rangle \\ + 2\varepsilon \left\langle D^2P(x-y)^2(q-\hat{q}), q-\hat{q} \right\rangle. \end{aligned}$$

Invoking (5.17), we conclude that

$$0 < \gamma \leq \alpha \left\langle D^2P(\hat{x}-\hat{y})(q-\hat{q}), q-\hat{q} \right\rangle + 2\varepsilon \left\langle D^2P(\hat{x}-\hat{y})^2(q-\hat{q}), q-\hat{q} \right\rangle. \tag{5.18}$$

If $J(DP(\hat{x}-\hat{y}))$ is a singleton, then $q = \hat{q}$ and we have a contradiction. To handle the case in which $J(DP(\hat{x}-\hat{y}))$ is not a singleton, we invoke (5.12) (ii). The first term above can then be dropped, as it is nonpositive, and we may let $\varepsilon \downarrow 0$ in the second term to reach our contradiction. Note that the points

$\hat{x}, \hat{y}, q, \hat{q}$ depend on α, ε. However, $\hat{x}, \hat{y} \in U$, and P is C^2, while $|q| = |\hat{q}| = 1$, so there is no difficulty in taking the limit.

We have almost proved:

Theorem 5.2. *Let there exist a penalty function P satisfying (5.12). Let U be bounded, $u, v \in C(\overline{U})$, $F^+(D^2u, Du) \geq 0$ and $F^-(D^2v, Dv) \leq 0$ in U. Then*

$$u(x) - v(x) \leq \max_{\partial U}(u - v) \text{ for } x \in U. \tag{5.19}$$

Proof. The remaining point to settle is this: what was argued above is that if u enjoys comparison with cones from above, then the u_ε of Section 2.1 has the desired properties. It was not shown, due to the flow of the proof we used, that if merely $F^+(D^2u, Du) \geq 0$, then u_ε has the desired properties. In this view, (5.19) has been proved if one of $u, -v$ enjoys comparison with cones from above. If $\overline{B}_r(y) \subset U$, the cone function

$$C(x) = u(y) + \frac{\max_{|z-y|=r} u(z) - u(y)}{r}|x - y|$$

enjoys comparison with cones in $B_r(y) \setminus y$, and further satisfies $u \leq C$ on the boundary of $B_r(y) \setminus \{y\}$. Hence, $u \leq C$ in $\overline{B}_r(y)$. This is the basic "comparison with cones property" needed of u to assure that the u_ε do all we required of them. □

The general program of the proof of Theorem 5.2, "shielding" the discontinuities by a suitable penalty function, is not new. See, for example, [8], Section 9, and the references at the end of the section. There are many later examples, from which we mention Ishii and Souganidis [10]. The task is to find suitable penalty functions.

We are unable to find a suitable P in general, but will show that one exists if $n = 2$ and the unit sphere S contains at most a finite number of line segments. Moreover, we will exhibit a suitable P in the cases when $|\cdot|$ is either of the norms

$$|x|_\infty = \max_{i \in \{1,\ldots,n\}} |x_i| \quad \text{or} \quad |x|_1 = \sum_{i=1}^{n} |x_i|. \tag{5.20}$$

These results complete the uniqueness theory for absolutely minimizing functions in the l_p norm on $\mathbb{R}^n$, $1 \leq p \leq \infty$.

5.2. **The Case of the Maximum Norm.** Here we construct a penalty function for $|\cdot|_\infty$ by the method of guessing. In this case

$$J(p_1, \cdots, p_n) = \{(q_1, \cdots, q_n) : q_j \in \text{sign}(p_j), j = 1, 2, \cdots, n\} \tag{5.21}$$

where

$$\operatorname{sign}(a) = \begin{cases} \{1\} & \text{if } a > 0, \\ [-1,1] & \text{if } a = 0, \\ \{-1\} & \text{if } a < 0. \end{cases} \tag{5.22}$$

If $J(p)$ is not a singleton, say $q, \hat{q} \in J(p)$, owing to the form (5.21) of J, the j^{th} component of $q - \hat{q}$ is zero if $p_j \neq 0$ and if $p_j = 0$, the j^{th} component of $q - \hat{q}$ might be any number in the interval $[-2, 2]$. If we choose

$$P(x) = x_1^4 + \cdots + x_n^4,$$

we win. From what was just said, if $p = DP(x) \neq 0$ and $q, \hat{q} \in J(DP(x))$, then $D^2P(x)$ is diagonal with its j^{th} diagonal element 0 whenever $q_j \neq \hat{q}_j$.

5.3. **Two Dimensions.** For more general norms, we need corresponding P's. Even if $n = 2$, constructing a suitable P does not seem to be a simple matter in general, and we will not do so in general. There are other methods to obtain the result described below in this case (see, eg, Section 5.4), but the one we present leads to an interesting appearance of the "1-Laplacian."

For any n, if $p \neq 0$ and $q, \hat{q} \in J(p)$, $q \neq \hat{q}$. Then the line segment $[q, \hat{q}]$ lies in the sphere S and

$$< p, q - \hat{q} >= 0.$$

Assume that $n = 2$ and that there are only finitely many nontrivial line segments in the sphere S, call them

$$L_1, \cdots, L_m.$$

Choose unit vectors $v_1, \cdots, v_m \in \mathbb{R}^2$ such that

$$\langle v_j, q - \hat{q} \rangle = 0 \text{ when } q, \hat{q} \in L_j. \tag{5.23}$$

What we want of P is this: writing elements of $\mathbb{R}^2$ as (x, y), for $j = 1, \ldots, m$, if

$$DP(x, y) \text{ is parallel to } v_j \tag{5.24}$$

then

$$\left\langle D^2P(x, y)(q - \hat{q}), q - \hat{q} \right\rangle \leq 0 \text{ for } (q - \hat{q}) \perp DP(x, y). \tag{5.25}$$

Now let n be general for a while. If A is an $n \times n$ symmetric matrix and $p \in \mathbb{R}^n \setminus \{0\}$, then $\langle Aq, q \rangle \leq 0$ for all $q \perp p$ iff

$$\langle ATx, Tx \rangle \leq 0 \text{ for all } x \in \mathbb{R}^n,$$

where T is the orthogonal projection on the hyperplane with normal p. That is, we must have

$$TAT \leq 0 \quad \text{where} \quad T = I - \hat{p} \otimes \hat{p}$$

and $\hat{p}$ is the (Euclidean) unit vector in the direction of p. We may as well use $T = |p|_2^2 I - p \otimes p$ in place of $I - \hat{p} \otimes \hat{p}$.

From here on we write $u(x_1, \ldots, x_n)$ instead of P. *Note* that u is not the u of previous sections. Taking $p = Du$ above and computing yields

$$T = \begin{bmatrix} {u_{x_2}}^2 + \cdots + {u_{x_n}}^2 & -u_{x_1}u_{x_2} & \cdots & -u_{x_1}u_{x_n} \\ -u_{x_1}u_{x_2} & {u_{x_1}}^2 + {u_{x_3}}^2 + \cdots + {u_{x_n}}^2 & \cdots & -u_{x_2}u_{x_n} \\ \vdots & \vdots & \cdots & \vdots \\ -u_{x_1}u_{x_n} & \cdots & -u_{x_n-1}u_{x_n} & {u_{x_1}}^2 + \cdots + {u_{x_n-1}}^2 \end{bmatrix} \tag{5.26}$$

which becomes, in the case $n = 2$,

$$T = \begin{bmatrix} {u_y}^2 & -u_x u_y \\ -u_x u_y & {u_{x_x}}^2 \end{bmatrix} \tag{5.27}$$

Computing further when $n = 2$

$$TD^2uT = (u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}) \begin{bmatrix} u_y^2 & -u_x u_y \\ -u_x u_y & u_x^2 \end{bmatrix} \tag{5.28}$$

the matrix on the right above is nonnegative, so TD^2uT is nonpositive if

$$\mathcal{L}[u] := u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} \le 0. \tag{5.29}$$

Recall that we only need to satisfy (5.29) when (5.24) holds, that is, when

$$(u_x, u_y) \text{ is parallel to } v_j \text{ for some } j,\ 1 \le j \le m. \tag{5.30}$$

Thrashing around, we are led to try u of the form

$$u(x, y) = r^a e^{U(\theta)}. \tag{5.31}$$

where r, θ are standard polar coordinates. *Note* that we will use U, V to denote functions below, not subsets of $\mathbb{R}^n$. A computation yields

$$\mathcal{L}(r^a e^{U(\theta)}) = r^{3a-4} a e^{3U(\theta)} \left(U'(\theta)^2 + aU''(\theta) + a^2\right) \tag{5.32}$$

and the sign of $\mathcal{L}u$ is the same as that of $U'(\theta)^2 + aU''(\theta) + a^2$. For the form (5.31) of u with $a = 4$, each condition (5.30) becomes a condition of the form

$$\alpha\,(4\cos(\theta) - \sin(\theta)U'(\theta)) + \beta\,(4\sin(\theta) + \cos(\theta)U'(\theta)) = 0 \tag{5.33}$$

where (α, β) is a unit vector perpendicular to some v_j. Rewrite this as

$$4(\alpha\cos(\theta) + \beta\sin(\theta)) + (\beta\cos(\theta) - \alpha\sin(\theta))U'(\theta) = 0. \tag{5.34}$$

Next choose ω so that

$$\cos(\omega) = \beta,\ \sin(\omega) = -\alpha$$

so that (5.34) becomes

$$4\sin(\theta - \omega) + \cos(\theta - \omega)U'(\theta) = 0,$$

we see that $\cos(\theta - \omega) = 0$ is ruled out and

$$U'(\theta) = -4\tan(\theta - \omega) \tag{5.35}$$

determines the arguments of U' we have to worry about. Each v_j in (5.30) contributes another ω_j, so our final conditions have the form

$$\begin{gathered} \text{if } U'(\theta) = -4\tan(\theta - \omega_j) \text{ for some } j, \\ \text{then } 4U''(\theta) + U'(\theta)^2 + 16 \le 0. \end{gathered} \tag{5.36}$$

It is easy to see (but not to explain) that there are such functions. We describe how to construct them informally. One designs a 2π periodic function U' with these properties by making U'' very negative at the places where $U'(\theta) = -4\tan(\theta - \omega_j)$ for some j. Letting $V = U'$, sketch the graphs of the $-4\tan(\theta - \omega_j)$, then choose a V whose integral over $[0, 2\pi]$ is zero and whose graph intersects that collection in a finite number of points. Then very locally adjust the slope of V at the points of crossing - without changing these points - so that $4V'(\theta) + V(\theta)^2 + 16 \le 0$ there in a manner which leaves V itself only slightly changed. Then adjust the values of V away from the crossing points so that the final result has mean value 0. Let $U' = V$.

5.4. **Smoothing the Norm: the l_1 Case.** We construct a penalty function for the case of the norm $|\cdot|_1$ of (5.20). The proof is an instance of a general idea, which we explain, then ignore thereafter. The idea is this: let ρ_ε be a standard mollifying kernel on $\mathbb{R}^n$. Set

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x - y)|y|\, dy,$$

choose ε suitably small, and then define $F(x) > 0$ for $x \neq 0$ by

$$f_\varepsilon\left(\frac{x}{F(x)}\right) = 1.$$

That is, F is the norm whose unit sphere is $\{x : f_\varepsilon(x) = 1\}$. Then put $P = F^4$. This method will work for the $|\cdot|_\infty$ case already treated more simply, and in the two dimensional case as well. We suspect that it works for general norms whose unit balls are polytopes. It also succeeds in other cases as well. Here we content ourselves with verifying it for the case of the l_1 norm, where everything becomes very explicit, without further reference to its origin.

The duality mapping in this case can be described as follows: set

$$I(p) = \{j \in \{1, 2, \ldots, n\} : |p_j| = |p|_\infty.\} \tag{5.37}$$

Then, for $p \neq 0$,

$$J(p) = \left\{ \sum_{j \in I(p)} \mu_j \mathrm{sign}(p_j) e_j : \mu_j \ge 0, \sum_{j \in I(p)} \mu_j = 1. \right\} \tag{5.38}$$

Let $l : \mathbb{R} \to \mathbb{R}$ be a smooth approximation to the absolute value function with the properties below.

$$\begin{aligned} &\text{(i) } l(s) = |s| \text{ for } s \in \mathbb{R}, |s| \geq \frac{1}{2n}, 0 \leq l(0), \\ &\text{(ii) } l \text{ is even and convex on } \mathbb{R}, \text{ and strictly increasing on } [0, \infty). \end{aligned} \tag{5.39}$$

Define $F : \mathbb{R}^n \to [0, \infty)$ by

$$f(x) = \sum_{i=1}^{n} l(x_i), \ f\left(\frac{x}{F(x)}\right) = 1, \tag{5.40}$$

or, more precisely,

$$\sum_{i=1}^{n} l\left(\frac{x_i}{F(x)}\right) = 1 \text{ if } x \neq 0 \text{ and } F(0) = 0. \tag{5.41}$$

The unique existence of F is guaranteed by the properties (5.39) and it is as smooth on $\mathbb{R}^n \setminus \{0\}$ as is l by the implicit function theorem. Moreover, $F(\lambda x) = \lambda F(x)$ for $\lambda \geq 0$; in fact, F is a norm, although we won't use this. Let e_i be the standard i^{th} coordinate vector of $\mathbb{R}^n$ for $i = 1, \ldots, n$. Computing the gradient DF of F, one finds

$$DF(x) = \kappa\left(\frac{x}{F(x)}\right) \sum_{i=1}^{n} l'\left(\frac{x_i}{F(x)}\right) e_i. \tag{5.42}$$

where

$$\kappa(y) = \frac{1}{\sum_{i=1}^{n} l'(y_i) y_i}. \tag{5.43}$$

Note that the denominator of κ vanishes only at the origin. We read off of (5.38) and (5.42), that $J(DF(x))$ contains more than one element if and only if

$$I(DF(x)) = \left\{ j \in \{1, 2, \ldots, n\} : \left| l'\left(\frac{x_j}{F(x)}\right) \right| = \left| \sum_{i=1}^{n} l'\left(\frac{x_i}{F(x)}\right) e_i \right|_{\infty} \right\} \tag{5.44}$$

contains more than one element. Moreover, then

$$J(DF(x)) = \left\{ q \in \mathbb{R}^n : q = \sum_{j \in I(x)} \mu_j \mathrm{sign}(x_j) e_j, \mu_j \geq 0, \sum_{j \in I(x)} \mu_j = 1 \right\}. \tag{5.45}$$

Note that $I(DF(x)) = I(x)$ to simplify writing below.

We claim that $j \in I(x)$ implies

$$l'\left(\frac{x_j}{F(x)}\right) = \mathrm{sign}(x_j) \text{ and } l''\left(\frac{x_j}{F(x)}\right) = 0. \tag{5.46}$$

Indeed, since $x/F(x) \in \{f = 1\}$, there is at least one j for which $l(x_j/F(x)) \geq 1/n$. By the properties (5.39), this implies that $|l'(x_j/F(x))| = 1$, which is as large as possible. Hence $|l'(x_j/F(x))| = 1$ for $j \in I(x)$. Moreover, $l' = \pm 1$ implies that $l'' = 0$, since ± 1 are extremal values of l'.

Suppose $q, \hat{q} \in J(DF(x))$. We claim that then

$$\frac{d^j}{dt^j} f\left(\frac{x}{F(x)} + t(q-\hat{q})\right)\Big|_{t=0} = 0 \text{ for } j = 1, 2. \tag{5.47}$$

Indeed, from (5.40), (5.45), (5.46), we find

$$\frac{d}{dt} f\left(\frac{x}{F(x)} + t(q-\hat{q})\right)\Big|_{t=0} = \sum_{j\in I(x)} l'\left(\frac{x_j}{F(x)}\right)(q_j - \hat{q}_j) = \sum_{j\in I(x)} \operatorname{sign}(x_j)(q_j - \hat{q}_j) = 0.$$

Next,

$$\frac{d^2}{dt^2} f\left(\frac{x}{F(x)} + t(q-\hat{q})\right)\Big|_{t=0} = \sum_{j\in I(x)} l''\left(\frac{x_i}{F(x)}\right)(q_j - \hat{q}_j)^2 = 0$$

because $l''(s) = 0$ whenever $l'(s) = \pm 1$. Thus (5.47) holds.

Set

$$g(\lambda, t) = f\left(\frac{x + F(x)t(q-\hat{q})}{\lambda}\right)$$

and notice that, via (5.47),

$$g(F(x), 0) = 1, \frac{\partial g}{\partial t}(F(x), 0) = \frac{\partial^2 g}{\partial t^2}(F(x), 0) = 0.$$

Therefore, solving $g(\lambda, t) = 1$ for $\lambda(t) = F(x + F(x)t(q-\hat{q}))$, we have

$$\frac{d}{dt} F(x + F(x)t(q-\hat{q}))\big|_{t=0} = \frac{d^2}{dt^2} F(x + F(x)t(q-\hat{q}))\big|_{t=0} = 0.$$

In particular, from the computation of the second derivative, we see that

$$F(x)^2 \left\langle D^2F(x)(q-\hat{q}), q-\hat{q}\right\rangle = 0, \text{ which implies } \left\langle D^2F(x)(q-\hat{q}), q-\hat{q}\right\rangle = 0,$$

as desired.

Finally we put

$$P(x) = F(x)^4$$

so that $P \in C^2(\mathbb{R}^n)$ and $DP(0) = 0$.

A calculation yields

$$DP(x) = 4F(x)^3 DF(x), \; D^2P(x) = 12F(x)^2 DF(x) \otimes DF(x) + 4F(x)^3 D^2F(x).$$

Thus $J(DP(x)) = J(DF(x))$. Suppose again that $q, \hat{q} \in J(DF(x))$. Then $\langle q, DF(x)\rangle = \langle \hat{q}, DF(x)\rangle = |DF(x)|_\infty$ and

$$\left\langle D^2P(x)(q-\hat{q}), q-\hat{q}\right\rangle = 12F(x)^2 \left\langle DF(x), q-\hat{q}\right\rangle^2 + 4F(x)^3 \left\langle D^2F(x)(q-\hat{q}), q-\hat{q}\right\rangle = 0.$$

We have verified everything.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, SANTA BARBARA, CA 93106

E-mail address: crandall@math.ucsb.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195

E-mail address: gunni@math.washington.edu

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETRIOT, MI

E-mail address: pywang@math.wayne.edu