Solutions

Homework 1

1.1 Verify that \( \langle x, y \rangle_k = nx_1y_1 + (n - 1)x_2y_2 + \ldots + x_ny_n \) defines an inner product on \( \mathbb{R}^n \).

Then compute the norm of \( u = (a_1, a_2, \ldots, a_n) \).

Proof. The given inner product is positive as

\[
\langle x, x \rangle_k = nx_1^2 + (n - 1)x_2^2 + \ldots + x_n^2 \geq 0,
\]

as it is a sum of non-negative numbers. Furthermore this inner product is definite as each term in the sum is non-negative, and if any term is positive the resulting sum will also be positive. Thus \( \langle x, x \rangle_k = 0 \) implies \( ix_i^2 = 0 \) for all \( i = 1, \ldots, n \). This, in turn, implies each \( x_i = 0 \), which implies the vector \( x \) has all zero entries or \( x = 0 \).

Additivity and homogeneity in the first slot follow from the distributive law, that is,

\[
\langle x + y, z \rangle_k = n(x_1 + y_1)z_1 + (n - 1)(x_2 + y_2)z_2 + \ldots + (x_n + y_n)z_n,
\]

\[
= nx_1z_1 + ny_1z_1 + (n - 1)x_2z_2 + (n - 1)y_2z_2 + \ldots + x_nz_n + y_nz_n,
\]

\[
= \langle x, z \rangle_k + \langle y, z \rangle_k,
\]

and

\[
\langle ax, y \rangle_k = nax_1y_1 + (n - 1)ax_2y_2 + \ldots + ax_ny_n,
\]

\[
= a(nx_1y_1 + (n - 1)x_2y_2 + \ldots + x_ny_n),
\]

\[
= a \langle x, y \rangle_k.
\]

Finally symmetry of the given inner product follows from the commutivity of multiplication in \( \mathbb{R} \).

\[
\langle x, y \rangle_k = nx_1y_1 + (n - 1)x_2y_2 + \ldots + x_ny_n,
\]

\[
= ny_1x_1 + (n - 1)y_2x_2 + \ldots + y_nx_n,
\]

\[
= \langle y, x \rangle_k.
\]

Therefore \( \langle x, y \rangle_k = nx_1y_1 + (n - 1)x_2y_2 + \ldots + x_ny_n \) defines an inner product on \( \mathbb{R}^n \). \( \square \)

1.2 Given real numbers \( a_{ij}, i, j = 1, 2 \), define \( \langle x, y \rangle = a_{11}x_1y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2 \), determine the conditions that make \( \langle x, y \rangle \) an inner product on \( \mathbb{R}^2 \).

Solution:

\[
\langle x, y \rangle = a_{11}x_1y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2
\]

determines an inner product on \( \mathbb{R}^2 \) if and only if the following conditions hold,

(a) \( a_{11} > 0 \),

(b) \( a_{11}a_{22} - a_{12}a_{21} > 0 \),

(c) \( a_{12} = a_{21} \).

Proof. (Necessity) First assume that \( \langle \cdot, \cdot \rangle \) defines an inner product, then by positivity and definiteness

\[
a_{11} = \langle (1, 0), (1, 0) \rangle > 0.
\]

Similarly,

\[
\langle (-a_{12}, a_{11}), (-a_{12}, a_{11}) \rangle = a_{11}a_{11}^2 - a_{12}a_{12}a_{11} - a_{21}a_{11}a_{12} + a_{22}a_{11}^2
\]

\[
= a_{11}(a_{11}a_{22} - a_{12}a_{21}) > 0.
\]
Since $a_{11} > 0$ this implies that $a_{11}a_{22} - a_{12}a_{21} > 0$. Finally, by symmetry

$$a_{12} = \langle (1,0), (0,1) \rangle = \langle (0,1), (1,0) \rangle = a_{21}.$$  

(Sufficiency) Next assume, conversely, that the real numbers $a_{ij}$ satisfy conditions a-c. Then positivity holds, as

$$\langle x, x \rangle = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2,$$

$$= a_{11} \left[ \left( x_1 + \frac{a_{12}}{a_{11}} x_2 \right)^2 - \left( \frac{a_{12}}{a_{11}} \right)^2 x_2^2 + \frac{a_{22}}{a_{11}} x_2^2 \right],$$

$$= a_{11} \left[ \left( x_1 + \frac{a_{12}}{a_{11}} x_2 \right)^2 + \left( \frac{a_{11}a_{22} - a_{12}^2}{a_{11}^2} \right) x_2^2 \right] \geq 0.$$  

Indeed, this also shows definiteness as the second term will only be zero when $x_2 = 0$, and if $x_2 = 0$ the first term can only be zero if $x_1 = 0$. Additivity holds as

$$\langle x + y, z \rangle = a_{11}(x_1 + y_1)z_1 + a_{12}(x_1 + y_1)z_2 + a_{12}(x_2 + y_2)z_1 + a_{22}(x_2 + y_2)z_2,$$

$$= a_{11}x_1z_1 + a_{11}y_1z_1 + a_{12}x_1z_2 + a_{12}y_1z_2 + a_{12}x_2z_1 + a_{12}y_2z_1 + a_{22}x_2z_2 + a_{22}y_2z_2,$$

$$= \langle x, z \rangle + \langle y, z \rangle.$$  

Homogeneity holds as

$$\langle cx, y \rangle = a_{11}cx_1y_1 + a_{12}cx_1y_2 + a_{12}cx_2y_1 + a_{22}cx_2y_2,$$

$$= c(a_{11}x_1y_1 + a_{12}x_1y_2 + a_{12}x_2y_1 + a_{22}x_2y_2),$$

$$= c\langle x, y \rangle.$$  

Finally symmetry also holds as

$$\langle x, y \rangle = a_{11}x_1y_1 + a_{12}x_1y_2 + a_{12}x_2y_1 + a_{22}x_2y_2,$$

$$= a_{11}y_1x_1 + a_{12}y_2x_1 + a_{12}y_1x_2 + a_{22}y_2x_2,$$

$$= \langle y, x \rangle.$$  

\[\square\]

1.3 Given an inner product on $\mathbb{R}^2$, compute $\|x - y\|$. Hence use the cosine law to deduce that $\langle x, y \rangle = \|x\|\|y\| \cos \theta$, with $\theta$ equal to the angle between $x$ and $y$, when $\langle x, y \rangle$ is the dot product.

Proof. First

$$\|x - y\| = \sqrt{\langle x - y, x - y \rangle},$$

$$= \sqrt{\langle x, x - y \rangle - \langle y, x - y \rangle},$$

$$= \sqrt{\langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle},$$

$$= \sqrt{\|x\|^2 - 2\langle x, y \rangle + \|y\|^2}.$$
Recall the law of cosines, which states that for a triangle with sides of length $a$, $b$, and $c$, with angle $\theta$ opposite to the side of length $c$ and between the sides of length $a$ and $b$ the following relation holds,

$$a^2 + b^2 - 2ab \cos \theta = c^2.$$ 

Now, for any two vectors, $x$ and $y$, the vector $x - y$, when starting at $y$ ends at $x$. This is encoded in the vector equation $y + (x - y) = x$. Thus $x$, $y$, and $x - y$ form a triangle. If $\theta$ is the angle between $x$ and $y$, then the law of cosines implies

$$\|x\|^2 + \|y\|^2 - 2\|x\||y\| \cos \theta = \|x - y\|^2.$$

Where $\|\cdot\|$ is the euclidean norm corresponding to the dot product as the inner product on $\mathbb{R}^n$. From the first computation this is the same as

$$\|x\|^2 + \|y\|^2 - 2\|x\||y\| \cos \theta = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2,$$

which, after cancellation, implies

$$\|x\||y\| \cos \theta = \langle x, y \rangle.$$

6.2 Suppose $u, v \in V$. Prove that $\langle u, v \rangle = 0$ if and only if

$$\|u\| \leq \|u + av\|$$

for all $a \in F$

Proof. (Necessity) Assume $\langle u, v \rangle = 0$ then for all $a \in F$

$$\|u + av\|^2 = \|u\|^2 + 2a\langle u, v \rangle + a^2\|v\|^2,$$

$$\geq \|u\|^2.$$ 

(Sufficiency) Assume $\|u\| \leq \|u + av\|$ for all $a \in F$. In particular, when $a = -\frac{\langle u, v \rangle}{\|v\|^2}$,

$$\|u\|^2 \leq \|u + av\|^2 = \|u\|^2 + 2a\langle u, v \rangle + a^2\|v\|^2,$$

$$= \|u\|^2 - 2\frac{\langle u, v \rangle}{\|v\|^2} \langle u, v \rangle + \left(\frac{\langle u, v \rangle}{\|v\|^2}\right)^2 \|v\|^2,$$

$$= \|u\|^2 - \left(\frac{\langle u, v \rangle}{\|v\|}\right)^2,$$

$$\leq \|u\|^2.$$

This, of course, implies that $\langle u, v \rangle = 0$. 

6.3 Prove that

$$\left(\sum_{j=1}^{n} a_j b_j \right)^2 \leq \left(\sum_{j=1}^{n} ja_j^2 \right) \left(\sum_{j=1}^{n} \frac{b_j^2}{j}\right)$$

for all real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$. 

Proof. The Euclidean inner product on $\mathbb{R}^n$ is given by,

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j,$$

for vectors $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. For $j = 1, \ldots, n$, let $\tilde{a}_j = \sqrt{j} a_j$, $\tilde{b}_j = b_j \sqrt{j}$, $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_n)$, and $\tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_n)$ then

$$\left( \sum_{j=1}^{n} a_j b_j \right)^2 = \left( \sum_{j=1}^{n} \tilde{a}_j \tilde{b}_j \right)^2 = \langle \tilde{a}, \tilde{b} \rangle^2,$$

and

$$\left( \sum_{j=1}^{n} j a_j^2 \right) \left( \sum_{j=1}^{n} \frac{b_j^2}{j} \right) = \left( \sum_{j=1}^{n} \tilde{a}_j^2 \right) \left( \sum_{j=1}^{n} \frac{\tilde{b}_j^2}{j} \right) = \|\tilde{a}\|^2 \|\tilde{b}\|^2.$$

This reduces the inequality to the Cauchy-Schwartz inequality. \qed

6.4 Suppose $u, v \in V$ are such that $\|u\| = 3, \|u + v\| = 4, \|u - v\| = 6$. What number must $\|v\|$ equal?

$$\|v\| = \sqrt{17}$$

Proof. By the Parallelogram Law

$$16 + 36 = 2(9 + \|v\|^2).$$

Thus $\|v\|^2 = 17$. \qed

6.5 Prove or disprove: there is an inner product on $\mathbb{R}^2$ such that the associated norm is given by

$$\|(x_1, x_2)\| = |x_1| + |x_2|.$$

Solution: There is no inner product on $\mathbb{R}^2$ with the given associated norm.

Proof. Consider the vectors in $\mathbb{R}^2$, $u = (1, 0)$ and $v = (0, 1)$. Then $u + v = (1, 1)$ and $u - v = (1, -1)$. With the given norm,

$$\|u + v\|^2 + \|u - v\|^2 = (1 + 1)^2 + (1 + 1)^2 = 8,$$

and

$$2(\|u\|^2 + \|v\|^2) = 2((1 + 0)^2 + (1 + 0)^2) = 4.$$

If the norm were associated to an inner product the Parallelogram Law would imply these two quantities must be equal. Since they are not, there can be no such inner product. \qed

6.6 Prove that if $V$ is a real inner-product space, then

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2}{4},$$

for all $u, v \in V$. 


Proof. Expand using bilinearity to get,
\[
\frac{\|u + v\|^2 - \|u - v\|^2}{4} = \frac{\langle u + v, u + v \rangle - \langle u - v, u - v \rangle}{4},
\]
\[
= \frac{\|u\|^2 + 2\langle u, v \rangle + \|v\|^2}{4} - \frac{\langle u \rangle^2 - 2\langle u, v \rangle + \|v\|^2}{4}.
\]
\[
\square
\]