7.1. Make $\mathcal{P}_2(\mathbb{R})$ into an inner-product space by defining
\[ \langle p, q \rangle = \int_0^1 p(x)q(x)dx. \]

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by $T(a_0 + a_1x + a_2x^2) = a_1x$.
(a) Show that $T$ is not self-adjoint.
(b) The matrix of $T$ with respect to the basis $(1, x, x^2)$ is
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

This matrix equals its conjugate transpose, even though $T$ is not self-adjoint. Explain why this is not a contradiction.

**Solution:**
(a) Consider the polynomials $p(x) = x$ and $q(x) = 1$. Then $T(x) = x$, while $T(1) = 0$. Therefore
\[ \langle T(x), 1 \rangle = \int_0^1 xdx = \frac{1}{2}, \]
while
\[ \langle x, T(1) \rangle = 0, \]
showing that $T$ is not self-adjoint.
(b) The basis $(1, x, x^2)$ is not orthonormal for this inner-product.

7.2. Prove or give a counterexample: the product of any two self-adjoint operators on a finite-dimensional inner-product space is self-adjoint.

**Solution:** The two matrices
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix},
\]
are self-adjoint. However
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix} = \begin{pmatrix}
0 & 2 \\
1 & 0
\end{pmatrix}
\]
is not self-adjoint.

7.3. Show that if $V$ is a real inner-product space, then the set of self-adjoint operators on $V$ is a subspace of $\mathcal{L}(V)$.

**Proof.** Let $T_1$ and $T_2$ be self-adjoint, then
\[ (T_1 + T_2)^* = T_1^* + T_2^* = T_1 + T_2, \]
showing that the set of self-adjoint operators is closed under addition. For any real number, $a$,
\[ (aT_1)^* = aT_1^* = aT_1, \]
showing that the set of self-adjoint operators is closed under scalar multiplication. Since the zero map is self-adjoint, this is a subspace of $\mathcal{L}(V)$. \[\square\]
7.8. Prove that there does not exist a self-adjoint operator \( T \in \mathcal{L}(\mathbb{R}^3) \) such that \( T(1, 2, 3) = (0, 0, 0) \) and \( T(2, 5, 7) = (2, 5, 7) \).

**Proof.** Every self-adjoint operator has orthogonal eigen-vectors. However, \((1, 2, 3)\) is an eginevector for \( T \) with eigenvalue 0 and \((2, 5, 7)\) is an eigen-vector for \( T \) with eigenvalue 1. However, \((1, 2, 3)\) and \((2, 5, 7)\) are not orthogonal, thus \( T \) cannot be self-adjoint. \( \square \)

7.15. Suppose \( U \) is a finite-dimensional real vector space and \( T \in \mathcal{L}(U) \). Prove that \( U \) has a basis consisting of eigenvectors of \( T \) if and only if there is an inner product on \( U \) that makes \( T \) into a self-adjoint operator.

**Proof.** Assume that \( U \) has a basis \((e_1, e_2, e_3)\) consisting of eigenvectors of \( T \), then define an inner-product on \( U \) by

\[
\langle a_1 e_1 + a_2 e_2 + a_3 e_3, b_1 e_1 + b_2 e_2 + b_3 e_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3.
\]

The basis \((e_1, e_2, e_3)\) is orthonormal for this inner-product, thus, by the spectral theorem, \( T \) is self-adjoint.

Conversely, if \( T \) is self-adjoint for some inner-product on \( U \), then the spectral theorem implies that there is an orthonormal basis of \( U \) consisting of eigenvectors of \( T \). \( \square \)

5.2.1. Consider \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( T(x) = Ax \) where \( A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Show that \( T^2 + I = 0 \). Compare this with Lemma 7.11.

**Solution:** Since \( A^2 = -I \), \((T^2 + I)(x) = A^2 x + x = 0\). Since \( \alpha = 0 \) and \( \beta = 1 \) satisfy \( \alpha^2 < 4\beta \), lemma 7.11 implies that for any self adjoint linear operator, \( T \), \( T^2 + I \) is invertible. Since the given map is not self adjoint, the lemma does not contradict the fact that \( T^2 + I = 0 \).

5.2.2. Consider \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( T(x) = Ax \) where \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Show that \( U = \{x_2 = 0\} \) is invariant under \( T \) but \( U^\perp \) is not.

**Solution:** Since \( U = \{(a, 0)\}, U^\perp = \{(0, b)\}, \)

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix},
\]

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix},
\]

5.3.1. Determine if the following matrices can be diagonalized.

\[
A_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{pmatrix},
\]
\[ A_2 = \begin{pmatrix}
 2 & 0 & 0 & 0 \\
 1 & 2 & 0 & 0 \\
 2 & 1 & 2 & 0 \\
 1 & 2 & 1 & 2 \\
 2 & 1 & 2 & 1 & 2
\end{pmatrix} . \]

**Solution:** The matrix \( A_1 \) is symmetric, thus by the spectral theorem \( \mathbb{R}^5 \) has an orthogonal basis consisting of eigenvectors for \( A_1 \). On the other hand the only eigenvalue for \( A_2 \) is 2, and \( \dim \text{null}(A_2 - 2I) = 1 \). Therefore there can be no basis of \( \mathbb{R}^5 \) consisting of eigenvectors for \( A_2 \), and it is not diagonalizable.

5.3.2. Let \( A \) be a symmetric real matrix and \( \alpha, \beta \) real numbers such that \( A^2 + \alpha A + \beta I = 0 \).

(a) Show that \( \alpha^2 \geq 4\beta \).

(b) Show that there are real numbers \( \lambda_1, \lambda_2 \) such that \((A - \lambda_1 I)(A - \lambda_2 I) = 0\).

(c) Deduce that one of the \( \lambda_1, \lambda_2 \) is an eigenvalue of \( A \).

**Solution:**

(a) For any vector \( x \in \mathbb{R}^n, x \neq 0, \)
\[ \|Ax\|^2 = (Ax) \cdot (Ax) = (A^T Ax) \cdot x = (A^2 x) \cdot x = -(\alpha Ax + \beta x) \cdot x. \]

Therefore,
\[ 0 = \|Ax\|^2 + \alpha (Ax) \cdot x + \beta \|x\|^2, \]
\[ 0 = \|Ax + \frac{\alpha}{2} x\|^2 + \left( \beta - \frac{\alpha^2}{4} \right) \|x\|^2, \]
\[ 4\beta - \alpha^2 = -4 \frac{\|Ax + \frac{\alpha}{2} x\|^2}{\|x\|^2} \leq 0. \]

This shows that \( \alpha^2 \geq 4\beta \).

(b) Let \( \lambda_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \) and \( \lambda_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} \) then
\[ (A - \lambda_1 I)(A - \lambda_2 I)x = A^2 x - (\lambda_1 + \lambda_2) Ax + \lambda_1 \lambda_2 x, \]
\[ = A^2 x - (-\alpha) Ax + \frac{1}{4}(\alpha^2 - (\alpha^2 - 4\beta))x, \]
\[ = A^2 x + \alpha Ax + \beta x, \]
\[ = 0. \]

(c) Since \((A - \lambda_1 I)(A - \lambda_2 I) = 0\), it must be that \( \text{range}(A - \lambda_2 I) \subset \text{null}(A - \lambda_1 I) \).

Thus if \((A - \lambda_1 I)\) is injective, then \((A - \lambda_2 I) = 0 \) and \( \lambda_2 \) is an eigenvalue for \( A \).

If \((A - \lambda_1 I)\) is not injective, then \( \lambda_1 \) is an eigenvalue of \( A \).